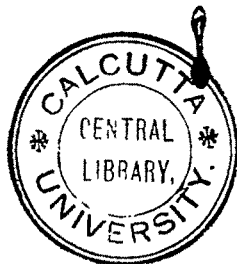


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READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY
PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF PUBLICATION

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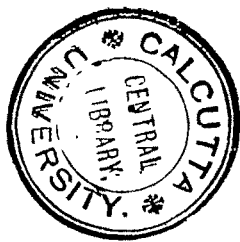
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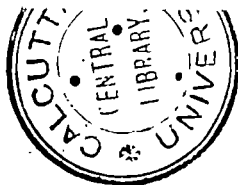
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INEQUALITIES OF A. MARKOFF AND S. BERNSTEIN FOR POLYNOMIALS AND RELATED FUNCTIONS¹

A. C. SCHAEFFER

Introduction. Weierstrass was the first to prove that an arbitrary continuous function which is defined over a closed finite interval may be uniformly approximated by a sequence of polynomials. The more difficult problem of best approximation by polynomials had earlier been initiated by Tchebycheff. A number of years later, in the early part of the present century, de la Vallée Poussin raised the following question of best approximation: Is it possible to approximate every polygonal line by polynomials of degree n with an error of $o(1/n)$ as n becomes large? (He had proved that the approximation can be carried out with an error of $O(1/n)$). This question was answered in the negative by Serge Bernstein in a prize-winning essay on problems of best approximation. In this paper Bernstein proved and made considerable use of an inequality concerning the derivatives of polynomials. This inequality and a related (and earlier) one by Andrew Markoff have been the starting point of a considerable literature. It has been found for example that the underlying ideas of these two inequalities are applicable to a much wider class of functions than polynomials. These inequalities have supplied one approach to questions concerning the derivatives of quasi-analytic functions. A generalization of Bernstein's theorem has been applied to almost periodic functions.

In discussing a mathematical theory we may emphasize either its applications or the salient points of the theory itself. The applications of Bernstein's inequality to problems of approximation (where it has probably had its greatest success) have been treated in the literature; see for example Dunham Jackson's book in the Colloquium Publications of the American Mathematical Society. On the other hand I am unaware of any recent résumé of the literature which has been suggested by the theorems of Markoff and Bernstein, so I shall discuss some of the investigations which have centered about these theorems.

Rational polynomials. It was the chemist Mendelejeff (author of the periodic table of chemistry) who asked the following question: If the bound of a rational polynomial over a given interval is known, how large may its derivative be in this interval? The maximum possible value of the derivative will of course depend on the degree of the poly-

¹ An address delivered before the Los Angeles meeting of the Society on November 23, 1940, by invitation of the Program Committee.

nomial. Actually Mendeleieff was interested in only the special case in which the polynomial is of degree two, and in this case the question is easily answered. The question for a polynomial of arbitrary degree is more difficult, but was answered in 1889 by A. Markoff [23] who proved the following theorem:

THEOREM I. *If $f(x) = \sum_{r=0}^n c_r x^r$ is a rational polynomial of degree n and $|f(x)| \leq 1$ in the interval $-1 \leq x \leq 1$ then in the same interval*

$$(1) \quad |f'(x)| \leq n^2.$$

The inequality of S. Bernstein [2] which appeared in his paper previously mentioned is as follows:

THEOREM II. *Under the conditions of Theorem I*

$$(2) \quad |f'(x)| \leq n(1-x^2)^{-1/2}, \quad -1 < x < 1.$$

Inequalities (1) and (2) are exact in the following sense. The constant n^2 in (1) cannot be replaced by any lower constant, in fact there is a well known polynomial which satisfies the conditions of Theorem I and whose derivative at the point $x=1$ is equal to n^2 . This polynomial is the n th Tchebycheff polynomial

$$(3) \quad T_n(x) = \cos(n \arccos x) = 2^{n-1} \prod_{j=1}^n \left\{ x - \cos\left((j-\frac{1}{2})\pi/n\right) \right\}.$$

It satisfies the differential equation

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0.$$

However, it is known that at points in the interior of the interval $(-1, 1)$ the derivative of $f(x)$ must be strictly less than n^2 . In Bernstein's theorem, on the other hand, the bound for $f'(x)$ is given as a function of x . This dominant $n(1-x^2)^{-1/2}$ is the least possible dominant only at special points in the interval $(-1, 1)$. It is, however, asymptotically equal to the precise bound at every fixed point in the interior of the interval as n becomes infinite.

It is easily seen that Bernstein's inequality gives a much lower estimate of $f'(x)$ than does Markoff's except for a small neighborhood of the points -1 and $+1$. This remark has been used to prove Markoff's theorem with the aid of Bernstein's. If x lies in the interval $|x| \leq \cos(\frac{1}{2}\pi/n)$ then (2) shows that

$$|f'(x)| \leq \frac{n}{\sin(\frac{1}{2}\pi/n)} \leq n^2.$$

Then to prove Markoff's inequality it is only necessary, in view of evident symmetry, to show that (1) is satisfied in the range $\cos(\frac{1}{2}\pi/n) \leq x \leq 1$, and this has been accomplished by several methods.²

The above remarks show that if x is a fixed point and n is increasing, the proper bound for $f'(x)$ is of order n when $|x| < 1$ and of order n^2 when $|x| = 1$. The reason that the order is different in these two cases can be seen from the following result due to Szegő [36]. Let Γ represent an open or a closed Jordan curve in the complex z -plane, and at any point z_0 on Γ let $\alpha\pi$ be the exterior angle. In the case in which Γ is an open curve there will of course be two exterior angles at each point z_0 except the ends. In this case let $\alpha\pi$ be the larger of the two exterior angles at z_0 . With suitable smoothness conditions on Γ we have the following theorem:

THEOREM III. *If $f(z) = \sum_0^n c_n z^n$ is a rational polynomial of degree n and $|f(z)| \leq 1$ on Γ then*

$$|f'(z_0)| < cn^\alpha, \quad z_0 \in \Gamma.$$

Here c is a constant which depends on z_0 and Γ , but not on n .

It is possible to show that the bound cn^α in this inequality is of the precise order as n becomes infinite. In the case in which Γ is the interval $(-1, 1)$ of the real axis we see that $\alpha=2$ at the endpoints of the interval, so this theorem shows that the derivative is of order n^2 . On the other hand at points in the interior of the interval $\alpha=1$, so at these points the derivative is of order n .

Theorem III can be proved by the use of conformal mapping. Let

$$\omega = \phi(z) = az + a_0 + a_1/z + \dots$$

be a function which maps the exterior of Γ onto the exterior of the unit circle in the ω plane in such a way that the points at infinity correspond. Then the function $f(z)\phi^{-n}(z)$ is regular in the exterior of Γ , even at infinity. Since it is bounded by 1 on Γ it has the same bound outside Γ ,

$$(4) \quad |f(z)| \leq |\phi(z)|^n.$$

Now the behavior of the mapping function $\phi(z)$ near the boundary of Γ has been effectively studied so it is possible to obtain good estimates of $f(z)$ from (4). To obtain an estimate for the derivative of $f(z)$ we write

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz$$

² See Kellogg [20], M. Riesz [29], Schur [32].

where C is a circle with center at z_0 and properly chosen radius. The theorem is then proved by taking absolute magnitudes under the integral sign and making use of (4).

In the case in which Γ is the segment $(-1, 1)$ of the real axis the mapping function $\phi(z)$ is especially simple. In this special case Montel [25] had earlier used a similar method to obtain inequalities (1) and (2), although not in their precise form. He showed, for example, that under the conditions of Theorem I the derivative of $f(x)$ is less than $4n^2$.

There are analogous inequalities for the Riemann-Liouville generalized derivatives of positive order α :

$$D^\alpha f(x) = \frac{1}{\Gamma(p-\alpha)} \frac{d^p}{dx^p} \int_{-1}^x (x-t)^{p-\alpha-1} f(t) dt$$

where $\alpha > 0$ and p is the integer part of $\alpha + 1$. Montel showed that if $f(x)$ satisfies the conditions of Theorem I then in every interval $(-x_0, x_0)$, $0 < x_0 < 1$,

$$|D^\alpha f(x)| \leq Kn^\alpha, \quad K = K(\alpha, x_0), \quad \alpha > 0.$$

This result was later generalized by Sewell [34] who obtained inequalities for the fractional derivatives of those polynomials which satisfy the conditions of Theorem III.

Trigonometric polynomials. An extension of Theorem II to trigonometric polynomials was found by Bernstein. By a trigonometric polynomial of degree n we mean an expression of the form

$$(5) \quad F(\theta) = \sum_0^n (a_r \cos r\theta + b_r \sin r\theta).$$

If in Theorem II we make the substitution $x = \cos \theta$, then, as is well known, $f(\cos \theta) = \sum_0^n c_r \cos^r \theta$ may be written as a trigonometric polynomial of degree n in which only the cosine terms appear. Since $d\theta = -(1-x^2)^{-1/2} dx$, inequality (2) states that

$$\left| \frac{d}{d\theta} f(\cos \theta) \right| \leq n.$$

More generally we may state the following proposition:

THEOREM IV. *If $F(\theta)$ is a trigonometric polynomial of degree n and $|F(\theta)| \leq 1$ for real θ , then*

$$(6) \quad |F'(\theta)| \leq n, \quad \theta \text{ real.}$$

This result was proved by Bernstein [2] at the same time as Theorem II, except that in (6) he had $2n$ in place of n . Inequality (6) in the present form first appeared in print in a paper of Fekete [15] who attributes the proof to Fejér. Bernstein [5, p. 39] attributes the proof to Landau. This theorem has a wide application to problems of approximation by trigonometric polynomials. The example $F(\theta) \equiv \sin n\theta$ shows that no lower constant than n will suffice in (6).

As already mentioned, under the substitution $x = \cos \theta$ a rational polynomial is transformed into a cosine polynomial in θ of the same degree; and the converse is also true. On the other hand, a sine polynomial $F(\theta) = \sum_{\nu=1}^n b_{\nu} \sin \nu\theta$ is transformed into a function of the form $(1-x^2)^{1/2} P_{n-1}(x)$ where $P_{n-1}(x)$ is a rational polynomial of degree $n-1$ in x . In this sense the trigonometric polynomials represent a wider class of functions than the rational polynomials. This distinction is of some importance because the class of trigonometric polynomials possesses a flexibility which makes several theorems easier to prove in terms of trigonometric polynomials than in terms of rational polynomials.

Bernstein's proof of Theorem IV was based on a variational method. Simpler proofs of this theorem have been obtained by M. Riesz, F. Riesz, and de la Vallée Poussin. Each of these methods has led to interesting extensions of Theorem IV, so we shall discuss them briefly. M. Riesz' proof is based on an important interpolation formula which expresses the derivative of a trigonometric polynomial in terms of the values of the polynomial at $2n$ equidistant points.

M. RIESZ' INTERPOLATION FORMULA [29]. *If $F(\theta)$ is a trigonometric polynomial of degree n then*

$$(7) \quad F'(\theta) = \sum_0^{2n-1} (-1)^r \rho_r F(\theta + \theta_r).$$

Here ρ_r and θ_r are constants which depend only on n and satisfy $\rho_r > 0$, $\sum \rho_r = n$, $0 < \theta_0 < \theta_1 < \dots < \theta_{2n-1} < 2\pi$.

Theorem IV is an obvious consequence of (7). This interpolation formula was used by Zygmund to obtain a generalization of Theorem IV. Zygmund [42] proved the following:

THEOREM V. *Let $p \geq 1$. If $F(\theta)$ is a trigonometric polynomial of degree n then*

$$(8) \quad \left\{ (2\pi)^{-1} \int_{-\pi}^{\pi} |F'(\theta)|^p d\theta \right\}^{1/p} \leq n \left\{ (2\pi)^{-1} \int_{-\pi}^{\pi} |F(\theta)|^p d\theta \right\}^{1/p}.$$

This result is proved by use of M. Riesz' interpolation formula and Jensen's inequality for convex functions. Theorem IV is obtained from (8) by letting p become infinite.

The proof of Theorem IV due to F. Riesz [28] begins with the representation

$$(9) \quad F'(\theta) = \frac{1}{\pi} \int_0^{2\pi} F(\theta + \phi) \left\{ \sum_1^n \nu \sin \nu \phi \right\} d\phi.$$

This may be obtained by use of the familiar integral representation of the coefficients a_ν and b_ν . The underlying idea in this method is the completion of the kernel, an idea which had earlier been used by Landau and by Fejér in other problems. The kernel in (9) is not positive, but since Γ is a trigonometric polynomial of degree n we may add to the kernel terms in $\cos \nu \phi$ and $\sin \nu \phi$ where ν is greater than n , and the value of the integral will not be changed. Adding a suitable trigonometric sum in (9) we obtain

$$(10) \quad F'(\theta) = \frac{1}{\pi} \int_0^{2\pi} F(\theta + \phi) \left\{ n + 2 \sum_1^n (n - \nu) \cos \nu \phi \right\} \sin n\phi d\phi.$$

The kernel in this case is more regular than before, although it is not positive. If we take absolute magnitudes under the integral sign and replace $|F|$ by 1 and $|\sin n\phi|$ by 1, what is left of the integrand is positive (it is essentially the Fejér kernel). Then we may integrate termwise and we obtain

$$|F'(\theta)| \leq 2n.$$

This is less precise than Theorem IV, but F. Riesz showed how the kernel could be completed still further, and Theorem IV could be proved by this method.

de la Vallée Poussin's proof [40] of Theorem IV is based on the well known theorem that a trigonometric polynomial of degree n can have at most $2n$ real zeros modulo 2π . If λ and α are real, $-1 < \lambda < 1$, the function

$$(11) \quad g(\theta) = \cos n\theta - \lambda F(\theta + \alpha)$$

is positive wherever $\cos n\theta = +1$ and is negative wherever $\cos n\theta = -1$. It therefore vanishes at $2n$ different points modulo 2π . It can be shown that if the derivative of $F(\theta)$ is greater than n , then λ and α can be so chosen that $g(\theta)$ has at least $2n+1$ zeros modulo 2π . This is impossible since $g(\theta)$ is a trigonometric polynomial of degree $2n$.

The method of de la Vallée Poussin applies only to polynomials

with real coefficients, but this is a restriction which is easily removed. For, if $F(\theta)$ is any polynomial which satisfies the conditions of Theorem IV and γ is a complex constant of unit modulus then $\gamma F(\theta)$ will satisfy the conditions of Theorem IV, and so will $\Re\{\gamma F(\theta)\}$. But γ can be so chosen that the derivative of $\gamma F(\theta)$ at any preassigned point is real, and then the derivative of $\Re\{\gamma F(\theta)\}$ will have the same modulus as $F'(\theta)$ at this point. The inequalities which have been mentioned until now hold whether the polynomials have real or complex coefficients, and this is true in all subsequent cases except where the contrary is stated.

de la Vallée Poussin's method can be used to show that under the conditions of Theorem IV we have the sharper inequality

$$(12) \quad n^2(F(\theta))^2 + (F'(\theta))^2 \leq n^2.$$

Here $F(\theta)$ is assumed real. In the general case in which the polynomial is complex we cannot say that the sum of the absolute magnitudes of the two terms on the left is less than n^2 . This is shown by the example $F(\theta) = e^{in\theta}$. Inequality (12) was first explicitly stated by van der Corput and Schaake [9], although it is implicit in an earlier inequality due to Szegő.

Polynomials in several variables. Now any trigonometric polynomial is the boundary function on the unit circle of a harmonic polynomial of the same degree. If $F(\theta)$ is given by (5) then

$$(13) \quad F(r, \theta) = \sum_0^n r^n (a_n \cos n\theta + b_n \sin n\theta)$$

is a harmonic polynomial of degree n and $F(1, \theta) = F(\theta)$. This harmonic function may also be expressed as a polynomial of degree n in the rectangular coordinates x and y under the substitution $x = r \cos \theta$, $y = r \sin \theta$; and it satisfies Laplace's equation $F_{xx} + F_{yy} = 0$.

Interpreting Theorem IV from the point of view of harmonic functions, it states that if a harmonic polynomial of degree n is bounded by 1 on the unit circle $r = 1$ then its tangential derivative on the unit circle is bounded by n . Under the same conditions, how large can the derivative be in other directions? Now the gradient of a function is a vector equal in magnitude to the largest directional derivative of the function. This question is therefore answered in the following theorem of Szegő [37].

THEOREM VI. *If $F(r, \theta)$ is a harmonic polynomial of degree n and $|F(1, \theta)| \leq 1$ then for $r \leq 1$*

$$(14) \quad |\text{grad } F(r, \theta)| \leq n.$$

To prove this theorem it is sufficient to show that (14) is satisfied on the unit circle, that is, when $r=1$. Also it is sufficient to show that for all real α the expression

$$(15) \quad \cos \alpha \frac{\partial F}{\partial r} + \sin \alpha \frac{\partial F}{\partial \theta}$$

is bounded by n . From this point the proof parallels the method of F. Riesz. We obtain by a familiar method of Fourier series an integral representation of (15) which is slightly more complicated than (9). The idea then is to complete the kernel.

At about the same time Kellogg investigated problems concerning the derivatives of general, not necessarily harmonic, polynomials in two and more variables. Suppose that $P(x, y)$ is a polynomial of degree n in rectangular variables, and that it is bounded by 1 in the unit circle $x^2 + y^2 \leq 1$. Here of course we cannot apply the maximum principle of harmonic functions so we assume that the polynomial is bounded in the closed interior of the unit circle, not merely on the boundary. The inequalities in this case will of course not be the same as those that are valid for harmonic polynomials. Along any diameter of the unit circle $P(x, y) = P(r \cos \theta, r \sin \theta)$ will be a polynomial of degree n in $r = (x^2 + y^2)^{1/2}$, where we allow r to take negative values. Theorems I and II then show that its derivative with respect to r (the derivative of P in the radial direction) will be dominated by the smaller of the two numbers $n(1-r^2)^{-1/2}$ and n^2 . Here we may ask: How large can the derivative be in other directions? Kellogg answered this by showing that the gradient of $P(x, y)$ is bounded by the smaller of the two numbers $n(1-x^2-y^2)^{-1/2}$ and n^2 . This includes both Theorem I and Theorem II, for if $f(x)$ satisfies the conditions of Theorem I then it may be considered as a polynomial in x and y , with the coefficients of terms in y equal to zero. As such it will be bounded by 1 in the unit circle, and Kellogg's result then shows that the derivative of $f(x)$ must satisfy inequalities (1) and (2).

There are analogous inequalities for polynomials in more than two variables. Kellogg [20] proved the following:

THEOREM VII. *If $P(x_1, x_2, \dots, x_m)$ is a polynomial of degree n which is bounded by 1 in the unit sphere $x_1^2 + x_2^2 + \dots + x_m^2 \leq 1$ then in the same sphere the gradient of P is dominated by the smaller of the two numbers $n(1-x_1^2-x_2^2-\dots-x_m^2)^{-1/2}$ and n^2 .*

Using a rotation of axes, this theorem in the case of more than two

variables can be deduced from the special case in which there are two variables.

Theorem VI also suggests problems concerning the gradient of polynomials in more than two variables. For harmonic polynomials of three variables which are bounded by 1 on the unit sphere the precise bounds of the gradient are known, except, curiously, when the polynomial is of degree two or three [38]. Unlike Theorem VII, the proper bound for the gradient of harmonic polynomials in three dimensions does not seem to be deducible from the corresponding inequality in the two-dimensional case. The proper bound for the gradient in the three-dimensional case is considerably more complicated than (14).

Entire functions. An extension of Theorem IV to a much wider class of functions than polynomials was noted by Bernstein. Clearly a trigonometric polynomial $F(z) = \sum_0^n (a_\nu \cos \nu z + b_\nu \sin \nu z)$ is periodic with period 2π , and is also an entire function of the complex variable z . It also belongs to the class of entire functions which are of exponential type, that is, those which satisfy, for some positive constant ρ ,

$$(16) \quad F(z) = O(e^{\rho|z|})$$

uniformly in every direction as $|z| \rightarrow \infty$. In the case in which $F(z)$ is a trigonometric polynomial, (16) is true with ρ equal to the degree of the polynomial. Bernstein [4] found that the condition in Theorem IV that $F(z)$ is a trigonometric polynomial can be replaced by the milder condition that it is an entire function of exponential type. He proved the following:

THEOREM VIII. *If $F(z)$, $z = x + iy$, is an entire function of exponential type which satisfies*

$$|F(z)| = O(e^{\rho|z|}), \quad |F(x)| \leq 1,$$

then on the real axis

$$(17) \quad |F'(x)| \leq \rho.$$

In this theorem ρ need not be an integer, and the functions which satisfy these conditions are not in general periodic. This theorem is applicable for example to the general Fourier-Stieltjes transform

$$(18) \quad F(z) = \int_{-\rho}^{\rho} e^{izt} d\alpha(t)$$

where $\alpha(t)$ is of bounded variation.

If $F(z)$ satisfies the conditions of Theorem VIII it can be shown by repeated use of the Phragmén-Lindelöf principle that

$$(19) \quad |F(x + iy)| \leq e^{\rho|y|}.$$

If x is a point on the real axis let C be a circle with radius $1/\rho$ and center at x , and write

$$F'(x) = \frac{1}{2\pi i} \int_C \frac{F(z)}{(z-x)^2} dz.$$

Then taking absolute magnitudes under the integral sign and making use of (19) we obtain

$$|F'(x)| \leq e\rho.$$

This is less precise than inequality (17). Inequality (17) can be proved using a modification of any of the methods outlined for proving Theorem IV.

Boas [7] obtained sharper results than in Theorem VIII, but under the slightly more stringent condition that $F(z)$ is representable in the form (18), and is bounded by 1 on the real axis.

Higher derivatives. The inequalities which have been mentioned until now have concerned the first derivative of the function. In some cases the proper bounds for the higher derivatives can be inferred by an inductive process from the inequalities for the first derivative. This is the case for example in Theorem IV. If $F(\theta)$ satisfies the conditions of that theorem then the function $F'(\theta)/n$ is a trigonometric polynomial of degree n , and is bounded by 1. Hence we infer from (6) that $|F''(\theta)| \leq n^2$. In the same way we obtain the best bounds for all higher derivatives of $F(\theta)$. This inductive process does not give the best bounds in all cases, however. If $f(x)$ is a rational polynomial of degree n and is bounded by 1 in $(-1, 1)$ then repeated use of Theorem I will show that $|f'(x)| \leq n^2(n-1)^2$. It is known that the true bound for $f'(x)$ is less than this. It was Wladimir Markoff (brother of A. Markoff) who found the proper bounds for the higher derivatives of $f(x)$ under these conditions. W. Markoff [24] proved the theorem which follows.

THEOREM IX. *Under the conditions of Theorem I*

$$(20) \quad |f^{(p)}(x)| \leq \frac{n^2(n^2-1^2)(n^2-2^2) \cdots (n^2-(p-1)^2)}{1 \cdot 3 \cdot 5 \cdots (2p-1)};$$

$$-1 \leq x \leq 1; p = 1, 2, \cdots, n.$$

The right-hand side of this inequality is exactly equal to $T_n^{(p)}(1)$, where $T_n(x)$ is the n th Tchebycheff polynomial (3). Markoff's proof of this result is based on a variational method. For fixed p suppose that $f(x)$ is an extremal polynomial, that is, suppose its p th derivative assumes the maximum possible value under the conditions of Theorem IX. (It is easily shown that this polynomial exists.) Markoff used a variational method to show that $|f(x)|$ must be equal to 1 at either n or $n+1$ different points in the interval $(-1, 1)$. In the latter case $\pm f(x)$ is the n th Tchebycheff polynomial, whose derivatives are easily shown to satisfy (20). In the former case it is possible to show that $f(x)$ satisfies a differential equation of the form

$$(21) \quad 1 - (f(x))^2 = \frac{(1-x^2)(x-b)(x-c)}{n^2(x-a)^2} (f'(x))^2.$$

Here a, b, c are real constants which depend upon one parameter. Markoff was then able to show that the derivatives of this class of polynomials satisfy (20), but the proof is quite difficult.

The differential equation (21) appeared in a paper of Tchebycheff [39] in 1854, but the polynomial solutions are often named for Zolotareff, who investigated their properties extensively at a later date. Zolotareff [41] showed that this differential equation admits a polynomial solution if and only if the constants a, b, c satisfy certain equations, which involve elliptic integrals. These elliptic integrals were not used, however, by W. Markoff in the proof of Theorem IX.

W. Markoff also investigated a more general problem: If $k_0, k_1, \dots, k_n$ are given constants and $f(x) = \sum_{r=0}^n c_r x^r$ satisfies the conditions of Theorem I, what is the precise bound for the linear form $\sum c_r k_r$? The best bound will of course depend upon the constants k_r , as well as upon the degree of the polynomial. By suitably choosing the constants k_r , the linear form can be made equal to any derivative of $f(x)$ at any preassigned point. However, this general problem was not carried so far as the special problem concerning derivatives of $f(x)$.

Duffin and Schaeffer found a shorter proof of Theorem IX. In addition they showed that for inequality (20) to be satisfied it is not necessary to assume that the polynomial is bounded over the entire interval $(-1, 1)$. They proved [13] the following fact:

THEOREM X. *If $f(x) = \sum_{r=0}^n c_r x^r$ is a rational polynomial of degree n and $|f(x)| \leq 1$ at the $n+1$ points where $(T_n(x))^2 = 1$, then*

$$|f^{(p)}(x)| \leq \frac{n^2(n^2-1^2) \cdots (n^2-(p-1)^2)}{1 \cdot 3 \cdot 5 \cdots (2p-1)}; \\ -1 \leq x \leq 1; p = 1, 2, \dots, n.$$

In the case of the first derivative ($p=1$) this is a generalization of Theorem I, but there is no analogous generalization of Theorem II, at least in the obvious form. It is possible to show by use of an interpolation formula that if $f(x)$ satisfies the conditions of Theorem X then the inequality $|f'(x)| \leq n(1-x^2)^{-1/2}$ will be satisfied at special points, but this inequality will not be valid throughout the entire interval $(-1, 1)$.

Quasi-analytic functions. The theorems of Markoff and Bernstein lead in a natural way to problems concerning the derivatives of a class of functions which are in general not analytic. One of the simplest results of this nature is the following lemma which is useful in the proof of certain Tauberian theorems: *If $f(x)$ is two times differentiable for $x>0$, and if as $x \rightarrow \infty$*

$$f(x) = o(1), \quad f''(x) = O(1),$$

then

$$f'(x) = o(1).$$

More general problems of this nature have been investigated by Hardy and Littlewood [16], Landau, and others. One problem which presents itself is:

If $f(x)$ is n times differentiable in a given interval I (finite or infinite), and if throughout I ,

$$|f(x)| \leq 1, \quad |f^{(n)}(x)| \leq 1,$$

what is the proper dominant of $f^{(k)}(x)$ over I , $0 < k < n$?

When I is a finite interval the question has not been answered except in special cases. Approximate results were obtained by Ore and Gorny. Ore [26] proved, for example, that under the above conditions the first derivative must satisfy

$$|f'(x)| \leq \frac{2(n-1)^2}{L} \left\{ 1 + \frac{L^n}{n!} \right\}$$

where L is the length of the interval I . His proof is based on Taylor's formula and makes use of A. Markoff's theorem. For a finite interval an exact solution was given by Landau in the special case $n=2$. Landau [22] also investigated problems in which the second derivative satisfies a one-sided condition. Suppose $f(x)$ is real and $|f(x)| \leq 1$, $f''(x) \geq \alpha$ throughout a finite interval I , where α is some constant. Of course if α is too large with respect to the length of the interval

there will be no function which satisfies these conditions. But if α is small enough, in particular if $\alpha \leq 0$, the conditions are consistent and it is possible to find a dominant for the first derivative of the function. This dominant varies throughout the interval, becoming infinite at the endpoints.

In the case in which I is an infinite interval and n is arbitrary, the precise bounds for $f^{(k)}(x)$, $0 < k < n$, were obtained by Kolmogoroff [21]. The bound for $f^{(k)}(x)$ which was obtained by Kolmogoroff is always less than $\pi/2$, but its precise form is fairly complicated, involving the ratio of two infinite series. This result has been applied to questions concerning quasi-analytic functions. Duffin and Schaeffer [12] altered the problem slightly, making stronger hypotheses and obtaining stronger results. They proved: *If the real function $f(x)$ is n times differentiable and if for $-\infty < x < \infty$*

$$(a) (f(x))^2 \leq 1,$$

$$(b) (f^{(n-1)}(x))^2 + (f^{(n)}(x))^2 \leq 1,$$

then inequality (b) is true when n is replaced by any smaller positive integer. It is possible to obtain Theorem VIII from this result or that of Kolmogoroff by a simple argument.

In conclusion, I wish to acknowledge my indebtedness to Dr. Duffin and Professor Szegő to each of whom I owe a great deal. I am also indebted to Professor Uspensky for several historical remarks.

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VALUE REGIONS FOR CONTINUED FRACTIONS¹

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1. **Introduction.** In a recent paper² we showed that if $a_2, a_3, a_4, \dots$ lie in or upon the parabola

$$(1.1) \quad |z| - \Re(z) = \frac{1}{2}$$

then the continued fraction

$$(1.2) \quad \frac{1}{1 + \frac{a_2}{1 + \frac{a_3}{1 + \frac{a_4}{1 + \dots}}}}$$

converges if and only if the series $\sum |b_n|$ diverges, where $b_1=1$, $a_n=1/b_{n-1}b_n$, ($n=2, 3, 4, \dots$), the series being considered as divergent if some a_n vanishes. Further, if z lies outside this parabola, the periodic continued fraction

$$(1.3) \quad \frac{1}{1 + \frac{\bar{z}}{1 + \frac{z}{1 + \frac{\bar{z}}{1 + \frac{z}{1 + \dots}}}}},$$

in which $\bar{z}$ is the conjugate of z , diverges, so that the parabola (1.1) is the "best" curve symmetrical with respect to the real axis.

The principal object of the present paper is to show that when $a_2, a_3, a_4, \dots$ lie in or upon the parabola (1.1) then all the approximants of (1.2) lie in or upon the circle

$$(1.4) \quad |z - 1| = 1.$$

If z_1 is any value of z not zero which is in or upon this circle, then there is a value z in or upon the parabola (1.1) such that the value of the continued fraction (1.3) is z_1 , and therefore the circular domain is the "best" domain.

2. **Fundamental lemma.** If we adopt the notation

$$v_n = \frac{1}{1 + \frac{a_{n+1}}{1 + \frac{a_{n+2}}{1 + \dots}}},$$

then

$$v_n = \frac{1}{1 + a_{n+1}v_{n+1}}, \quad n = 1, 2, 3, \dots$$

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² W. T. Scott and H. S. Wall, *A converge theorem for continued fractions*, Transactions of this Society, vol. 47 (1940), pp. 155-172, p. 166. We refer to this paper later as CT.

Thus the continued fraction (1.2) may be regarded as a succession of linear transformations of elements v_n belonging to a set V , where the transformations depend upon elements a_n of a set U . This notion leads us to formulate the following lemma.

LEMMA. *If there exist in the complex plane sets of points U and V such that*

(i) $1/(1+u) \in V$ when $u \in U$,

(ii) $1/(1+uv) \in V$ when $u \in U, v \in V$,

then every approximant of the continued fraction (1.2), beginning with the second, lies in V when the elements $a_2, a_3, a_4, \dots$ lie in U .

Let A_n/B_n denote the n th approximant of (1.2). Then, by (i), $A_2/B_2 = 1/(1+a_2) \in V$ when $a_2 \in U$. Suppose now that we have verified that $A_n/B_n \in V$ when the a_n 's are in U . Then

$$A_{n+1}/B_{n+1} = 1/(1 + a_{n+1}v),$$

where $v = 1/1 + a_3/1 + a_4/1 + \dots + a_{n+1}/1$, and is in V if $a_3, a_4, \dots, a_{n+1}$ are in U by our assumption. Hence, by (ii), A_{n+1}/B_{n+1} is in V when $a_2, a_3, \dots, a_{n+1}$ are in U . This proves the lemma by induction.

We shall call the set U an *element region* for the continued fraction (1.2). As $a_2, a_3, a_4, \dots$ range over U , the values of the approximants of (1.2), and the values of (1.2), when convergent, constitute a set V which we call *the value region* corresponding to the element region U . These sets U and V satisfy the conditions of the lemma.

The element region U is called a *convergence region* for (1.2) if this continued fraction remains convergent when the a_n 's vary independently over U . A convergence region is necessarily bounded, for otherwise the a_n 's could be so chosen that the series $\sum |b_n|$ would converge, which implies divergence of the continued fraction by oscillation.

In the next section we shall obtain the value region for (1.2) when the element region is the set of points in and upon the parabola given by (1.1).

3. **The parabola-circle theorem.** *If $a_2, a_3, a_4, \dots$ lie within or upon the parabola (1.1) then all the approximants of the continued fraction (1.2) lie within or upon the circle $|z-1|=1$. This circular region with the point $z=0$ removed is the value region corresponding to the parabolic element region.*

We seek first a region V , not necessarily the value region, corresponding to the parabolic element region U , such that U and V satisfy the conditions of the lemma of §2. If $|z| - \Re(z) \leq \frac{1}{2}$, the continued fraction (1.3) converges to the value

$$w = \frac{-1 + 2iy + (4x + 1 - 4y^2)^{1/2}}{2(x + iy)}$$

where $z = x + iy$. If $w = \xi + i\eta$, then

$$(3.1) \quad \xi = \frac{1}{2x + 1}, \quad \eta = \frac{2y}{2x + 1},$$

when $|z| - \Re(z) = \frac{1}{2}$. Thus w traverses the circle $|w - 1| = 1$ as z traverses the parabola $|z| - \Re(z) = \frac{1}{2}$. From the nature of this mapping of the parabola into the circle it is clear that as z ranges over the interior of the parabola, w ranges over the whole interior of the circle. The value $w = 0$ is not assumed for a finite value of z . These considerations show that the region V which we seek must contain the region

$$(3.2) \quad |z - 1| \leq 1, \quad z \neq 0.$$

We shall show that the conditions of the lemma are satisfied when U is the parabolic region and V is the region (3.2).

If $u \in U$, $v \in V$, $u = \xi + i\eta$, $v = x + iy$ we have

$$(x - 1)^2 + y^2 \leq 1, \quad \eta^2 \leq \xi + \frac{1}{4}.$$

Let $v_1 = 1/(1 + uv) = p + iq$. Then the condition that v_1 shall lie in V , namely: $(p - 1)^2 + q^2 \leq 1$, is seen to reduce to the condition

$$(3.3) \quad x\xi - y\eta \geq -\frac{1}{2}.$$

The form of this inequality suggests that we examine the polars of the point $\xi + i\eta$ relative to the hyperbola $x^2 - y^2 = -\frac{1}{2}$ when $\eta^2 = \xi + \frac{1}{4}$. These polars form the one-parameter family of lines

$$(3.4) \quad x(\eta^2 - \frac{1}{4}) - y\eta = -\frac{1}{2}, \quad -\infty < \eta < +\infty.$$

The envelope of this family is the circle $(x - 1)^2 + y^2 = 1$. Thus the line $x\xi - y\eta = -\frac{1}{2}$ is tangent to the circle if $\xi + i\eta$ is on the parabola. One can easily see that if $\xi + i\eta$ is inside the parabola this line does not cut the circle. Hence if $\xi + i\eta$ is in the parabola, and $x + iy$ in the circle, the points $x + iy$ and the origin lie on one side of this line. It follows that the inequality (3.3) holds.

We have proved that U and V , as defined, satisfy the condition (ii) of the lemma. Since $v = 1$ is in V , it follows that (i) holds.

From the way in which we arrived at (3.2) as the locus of values of the continued fraction (1.3) as z ranges over the parabola, and from the observation that the value 0 cannot be assumed by (1.2) or any of its approximants when the a_n 's lie in the parabola, we now con-

clude that (3.2) is the value region corresponding to the parabolic element region. Our parabola-circle theorem is now completely proved.

It is interesting to note that the point of tangency of the line $x\xi - y\eta = -\frac{1}{2}$ to the circle $(x-1)^2 + y^2 = 1$ when $\xi + i\eta$ is on the parabola is given by

$$x = 1/(2\xi + 1), \quad y = 2\eta/(2\xi + 1).$$

Hence we have again the projective mapping (3.1).

4. Circular bounds for the value of a continued fraction. Instead of allowing $a_2, a_3, a_4, \dots$ to vary *independently* over the element region U , one may suppose certain of the a_n 's fixed in U and inquire as to the region V in which the value of the continued fraction must lie when it converges. In this way one may, for example, obtain an estimate for the error committed in using a certain approximant instead of the value of the continued fraction. We shall prove the following theorem.

THEOREM 4.1. *If $a_2, a_3, a_4, \dots$ lie within or upon the parabola (1.1), then the sequence of even approximants of (1.2), which is necessarily convergent, has a limit w which satisfies the inequality*

$$(4.1) \quad \left| w - \frac{\bar{a}}{a\bar{a} - b^2} \right| \leq \frac{b}{a\bar{a} - b^2},$$

where

$$a = 1 + a_2 \left\{ 1 - \frac{a_3}{1 + a_2 + a_4} \cdot \frac{|1 + a_3 + a_4|^2}{|1 + a_2 + a_4|^2 - |a_4|^2} \right\},$$

$$b = \frac{|a_2 a_3 a_4|}{|1 + a_2 + a_4|^2 - |a_4|^2}.$$

Put $a_n = u_n + iv_n$. Since, by hypothesis, $|a_n| - u_n \leq \frac{1}{2}$, we may write $|a_n| = u_n + h_n/2$, where $0 \leq h_n \leq 1$. Hence

$$\begin{aligned} |1 + a_2| &\geq 1 + u_2 \geq u_2 + (1 + h_2)/2 = |a_2| + \frac{1}{2}, \\ |1 + a_2 + a_3| &\geq 1 + u_2 + u_3 \\ &= (1 - h_2/2 - h_3/2) + |a_2| + |a_3| > |a_2|, \\ |1 + a_n + a_{n+1}| &\geq 1 + u_n + u_{n+1} \geq u_n + h_n/2 + h_{n+1} + h_{n+1}/2 \\ &\geq |a_n| + |a_{n+1}|, \quad n = 3, 4, 5, \dots \end{aligned}$$

If then we put

$$g_n = \frac{|1 + a_{2n+1} + a_{2n+2}| - |a_{2n+2}|}{|1 + a_{2n+1} + a_{2n+2}|}, \quad n = 1, 2, 3, 4, \dots,$$

we see that $0 < g_n \leq 1$, ($n=1, 2, 3, \dots$); and that

$$(1 - g_{n-1})g_n \geq \frac{|a_{2n}a_{2n+1}|}{|(1 + a_{2n-1} + a_{2n})(1 + a_{2n+1} + a_{2n+2})|},$$

$$n = 2, 3, 4, \dots,$$

$$g_1 \geq |a_3| / |1 + a_3 + a_4|.$$

Define $x_1, x_2, x_3, \dots$ by the equations

$$(4.2) \quad \frac{-a_2a_3}{(1 + a_2)(1 + a_3 + a_4)} = g_1x_1,$$

$$\frac{-a_{2n}a_{2n+1}}{(1 + a_{2n-1} + a_{2n})(1 + a_{2n+1} + a_{2n+2})} = (1 - g_{n-1})g_nx_n$$

($n=2, 3, 4, \dots$), so that $|x_1| \leq |a_2| / |1 + a_2| \leq 1 - (1/2|1 + a_2|) < 1$;
 $|x_n| \leq 1$, ($n=2, 3, 4, \dots$).

Now the even part³ of (1.2) is

$$\frac{1}{1 + a_2} - \frac{a_2a_3}{1 + a_3 + a_4} - \frac{a_4a_5}{1 + a_5 + a_6} - \dots$$

On making an equivalence transformation and introducing the expressions in (4.2) we find that this takes the form

$$(4.3) \quad \frac{1}{1 + a_2} \left\{ \frac{1}{1 + \frac{g_1x_1}{1}} + \frac{(1 - g_1)g_2x_2}{1} + \dots \right\}.$$

In case some a_n vanishes, the continued fraction terminates.

Now put

$$z = \frac{g_1}{1 + \frac{(1 - g_1)g_2x_2}{1}} + \frac{(1 - g_2)g_3x_3}{1} + \dots,$$

so that $w = 1/(1 + a_2)(1 + x_1z)$. For z we have the inequality⁴

$$\left| z - \frac{1}{2 - g_1} \right| \leq \frac{1 - g_1}{2 - g_1},$$

and consequently

$$\left| \frac{1}{w} - a \right| \leq b,$$

³ CT, p. 160.

⁴ H. S. Wall, *A class of functions bounded in the unit circle*, Duke Mathematical Journal, vol. 7 (1940), pp. 146-153, p. 148.

where a and b are as defined in the statement of the theorem. From this (4.1) now follows, and the theorem is proved.

Examples. If $a_2 = 2 + 3i/2$, $a_3 = -i/2$, $a_4 = -1/4$ we find that

$$\left| w - \frac{24(78 - 53i)}{8893} \right| \leq \frac{240}{8893}.$$

If $a_2 = 1/2$, $a_3 = 1/6$, $a_4 = 1/3$, we find that

$$|w - .69375| \leq .00625.$$

In this case the first four partial quotients are the same as those in a continued fraction for $\log 2 = .6931 \dots$

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SUR LE NOMBRE COMPLEXE BINAIRE

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I. INTRODUCTION

L'objet de ce travail est de démontrer que la théorie des fonctions polygènes¹ d'un variable complexe connu et celles d'un variable complexe dual et hyperbolique développées par les docteurs Vignaux et Durafona y Vedia² sont des cas particuliers d'une même théorie qui en contient d'autres.

On doit l'origine de cette théorie à la suivante interprétation géométrique de la définition d'unité imaginaire $i^2 = -1$, $j^2 = 1$, $k^2 = 0$.

Si nous considérons le point représentatif de cette unité, nous observons que son carré représente un autre point qui géométriquement signifie, dans le champ complexe ordinaire, une rotation de $+\pi/2$, et, dans le champ complexe hyperbolique, une rotation de $-\pi/2$ et dans le dual une translation à zéro.

Il n'y a rien de plus naturel que de considérer ces questions comme cas particuliers d'une rotation et d'une translation combinées, c'est-à-dire que nous mettons un complexe $a + b\alpha$ dont l'unité imaginaire α est telle que $\alpha^2 = \mu + \nu\alpha$, ce qui analytiquement exprime le concept géométrique que nous venons de dire. Quand

¹ Le mot polygène était introduit par E. Kasner. Voyez son premier papier: *A new theory of polygenic or nonmonogenic functions*, Science, vol. 66 (1927), pp. 581-582.

² *Sobre las funciones de una variable compleja hiperbólica*, Contribución al Estudio de las Ciencias Fisicomatemáticas, vol. 1, estudio 2^a, 1935. J. C. Vignaux, *Sobre la teoría de funciones poligenas de una y varias variables complejas duales*, ibid., estudio 3^a.

$\nu=0, \mu=-1$ nous avons le complexe ordinaire,
 $\nu=0, \mu=1$, nous avons le complexe hyperbolique,
 $\nu=0, \mu=0$, nous avons le complexe duel.

Alors on définit ainsi les opérations fondamentales: Égalité et addition comme dans le cas du variable complexe ordinaire. Produit

$$(a, b) \times (a', b') = (aa' + \mu bb', a'b + ab' + \nu bb').$$

Dans ce champ on pourrait entendre que le nombre μ joue le rôle de dilatation (puisque si $\nu=0$, on passe des complexes connus aux complexes généraux en multipliant par μ) et ν joue le rôle de rotation puisque il transporte le point correspondant au carré de l'unité imaginaire du champ réel au complexe.

Les propriétés de ce champ sont caractérisées par le déterminant

$$\Delta = \nu^2 + 4\mu = \begin{vmatrix} \nu & \mu \\ -4 & \nu \end{vmatrix}.$$

Selon que $\Delta < 0$, $\Delta = 0$, $\Delta > 0$ nous avons les complexes elliptiques, paraboliques ou hyperboliques, avec l'intéressante propriété que, avec un changement d'unités, ces champs peuvent être transformés respectivement en d'autres ayant la nouvelle unité $\alpha^2 = -1, 0, 1$.

Géométriquement ce changement d'unités signifie une rotation de l'axe y de sorte qu'il forme avec l'axe x un angle ω donné par $\operatorname{tg} \omega = -2/\nu$ et les formules de passage sont pour les champs elliptiques et hyperboliques $x = x' + p\nu y'$, $y = -2py'$; et pour le champ parabolique

$$x = x' - \frac{\nu b}{2} y', \quad y = by',$$

p étant une constante qui dépend du champ.

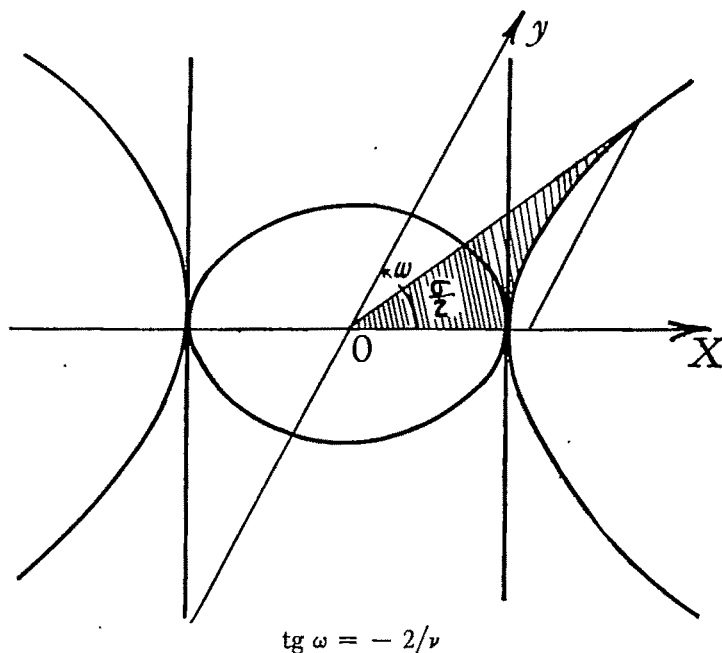
Nous introduisons le concept de *distance binaire* qu'on exprime pour un complexe $a + b\alpha$ à l'égard de l'origine des coordonnées

$$(1) \quad (|a^2 + \nu ab - \mu b^2|)^{1/2}.$$

Cette forme est invariable à l'égard des mouvements de la géométrie qui correspond à ces complexes et elle est aussi égale à la racine carrée du produit de deux complexes symétriques que nous définissons de la façon suivante: $a + b\alpha$, $(a + \nu b) - b\alpha$.

Cette formule (1) nous permet d'introduire le concept de module, de manière qu'on peut conserver la loi du module du produit des complexes, comme nous allons voir en introduisant l'expression trigonométrique des nombres de ce champ.

A ce propos nous introduisons le concept généralisé de fonctions trigonométriques, avec la définition suivante.



$$\operatorname{tg} \omega = -2/\nu$$

Nous considérons une conique avec centre, une ellipse pour le champ complexe ordinaire, une paire de droites parallèles pour le champ complexe parabolique et une hyperbole pour le champ complexe hyperbolique référées aux axes indiquées dans la figure.

En suivant la définition classique nous appellerons argument σ l'aire déterminée par la conique comme indiqué dans la figure, et nous appellerons $\sin \sigma$ ($\cos \sigma$) l'ordonnée (abscisse) du point de la conique déterminé par l'intersection avec le rayon vecteur et $\operatorname{tg} \sigma$ le quotient du $\sin \sigma$ et $\cos \sigma$.

On déduit à l'égard de ces définitions, entre autres, les formules suivantes:

$$\sin (\sigma_1 + \sigma_2) = \cos \sigma_1 \sin \sigma_2 + \cos \sigma_2 \sin \sigma_1 + \nu \sin \sigma_1 \sin \sigma_2,$$

$$\cos (\sigma_1 + \sigma_2) = \cos \sigma_1 \cos \sigma_2 + \mu \sin \sigma_1 \sin \sigma_2,$$

$$\sin (-\sigma_1) = -\sin \sigma_1, \quad \cos (-\sigma) = \cos \sigma + \nu \sin \sigma,$$

$$\operatorname{tg} (\sigma_1 + \sigma_2) = \frac{\operatorname{tg} \sigma_1 + \operatorname{tg} \sigma_2 + \nu \operatorname{tg} \sigma_1 \operatorname{tg} \sigma_2}{1 + \mu \operatorname{tg} \sigma_1 \operatorname{tg} \sigma_2},$$

$$\cos^2 \sigma + \nu \cos \sigma \sin \sigma - \mu \sin^2 \sigma = 1.$$

Ces formules contiennent comme cas particuliers les formules trigonométriques circulaires, hyperboliques et le dual introduit par le Dr. Vignaux.

En outre on trouve une généralisation des formules d'Euler :

$$\sin \sigma = \frac{e^{\alpha\sigma} - e^{-\alpha\sigma}}{2\alpha - \nu}; \quad \cos \sigma = \frac{\alpha(e^{\alpha\sigma} + e^{-\alpha\sigma}) - \nu e^{\alpha\sigma}}{2\alpha - \nu}.$$

Nous pouvons exprimer, maintenant, le nombre complexe binaire en forme trigonométrique $\rho(\cos \sigma + \alpha \sin \sigma)$ et on conserve dans cette formule les lois du produit, quotient, la formule de Moivre, et cetera.

Comme résultat de la définition de distance, on étend immédiatement le concept de voisinage, qui devient une ellipse, une paire de droites parallèles ou une hyperbole dont le centre est le point donné, respectivement, pour les champs complexe elliptique, parabolique ou hyperbolique.

Il est très important de remarquer que les asymptotes de ces coniques qui déterminent le voisinage du point zéro sont le lieu géométrique des diviseurs de zéro; ensuite de notre point de vue général les complexes elliptiques ont aussi des diviseurs de zéro, mais situés sur deux droites de pente imaginaire, qui sont les asymptotes imaginaires de l'ellipse.

Si le voisinage est conçu de cette façon, nous avons alors d'autres considérations que nous exprimons tout de suite.

Quand le rayon s'approche à zéro, la conique tend à se confondre avec ses asymptotes. Cette circonstance ainsi que celle où les diviseurs de zéro occupent les asymptotes, c'est-à-dire que la *distance binaire* entre ses points quelconques est zéro, donnent lieu à un intéressant concept de point, en le concevant comme un système formé par un point dans le sens ordinaire et la paire de droites parallèles aux droites isotropes (droites des diviseurs de zéro) du champ, qui passent par ce point.

De cette façon nous obtiendrons une image géométrique des points que nous appellerons *point elliptique*, *point parabolique* et *point hyperbolique*, selon le cas.

Nous considérons deux plans, l'un supérieur et l'autre inférieur, que nous imaginons unis comme des plans ponctuels, mais non pas comme des plans réglés, en exceptant une direction.

Alors le *point elliptique* sera représenté par un point et la paire de droites parallèles aux droites isotropes qui passent par ce point dans le plan inférieur; le *point parabolique* sera représenté par un point et la paire de droites coïncidentes selon la direction commune aux plans;

et point *hyperbolique* sera le point et les droites parallèles aux isotropes qui passent par ce point dans le plan supérieur.

De cette façon le plan inférieur contient les droites imaginaires et le supérieur les droites réelles.

En outre on peut passer, par continuité, en tournant autour d'un point, du plan supérieur à l'inférieur et réciproquement.

A ce passage géométrique continu on fait correspondre la variation continue du déterminant du champ, $\Delta = \nu^2 + 4\mu$, de valeurs positives à négatives.

II. FONCTIONS POLYGÈNES DE VARIABLE COMPLEXE BINAIRE

Les concepts de variable complexe, fonction, limite et continuité sont les classiques.

Dérivé radial. Soit la fonction de variable complexe binaire $w = f(z) = u(xy) + v(xy)\alpha$ où $z = x + y\alpha$ ($\alpha^2 = \mu + \nu\alpha$) et considérons le rapport $[f(z) - f(z_0)]/[z - z_0]$ avec z et z_0 points d'un domaine D de la fonction, avec z_0 fixe et z variable de telle façon qu'il n'appartient pas aux droites isotropes qui passent par z_0 .

Aussi nous avons

$$f(z) - f(z_0) = \Delta w = \Delta u + \Delta v \cdot \alpha, \quad z - z_0 = \Delta z = \Delta x + \Delta y \cdot \alpha$$

et

$$(1) \quad \frac{\Delta w}{\Delta z} = \frac{\Delta u + \Delta v \cdot \alpha}{\Delta x + \Delta y \cdot \alpha}.$$

Si les fonctions u et v sont différentielles dans le sens de Stolz-Fréchet il en résulte

$$\begin{aligned} \Delta u &= \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \\ \Delta v &= \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \epsilon'_1 \Delta x + \epsilon'_2 \Delta y, \end{aligned}$$

$\epsilon_1, \epsilon_2, \epsilon'_1, \epsilon'_2$ étant infiniment petits avec Δy et Δx .

Substituant en (1), divisant par Δx et passant au limite par $|\Delta z| \rightarrow 0$ on a

$$(2) \quad \left(\frac{dw}{dz} \right)_n = \frac{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \alpha \right) + n \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \alpha \right)}{1 + n\alpha} = X + Y\alpha$$

où $\Delta y / \Delta x = n$; expression que définit le dérivé radial selon la direc-

tion n . On doit exclure les directions $(\nu \pm m)/2\mu$ qui correspondent aux droites des diviseurs de zéro.

Conditions de monogénéité. La fonction w sera monogène dans un point quand le dérivé est égal pour une direction quelconque. Nous pouvons trouver les conditions de monogénéité en appliquant le théorème suivant. L'expression $(A+Bn)/(C+Dn)$, avec A, B, C et D complexes binaires et n un paramètre réel, est indépendante de n quand

$$AD = \bar{B}C,$$

tandis que C et D ne sont pas des diviseurs de zéro et n ne prennent pas deux valeurs qui fassent le dénominateur diviseur de zéro.

En appliquant ce théorème à notre expression (2) où

$$A = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \alpha, \quad B = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \alpha, \quad C = 1, \quad D = \alpha$$

il résulte, excepté les valeurs de $n = (\nu \pm m)/2\mu$, que la condition nécessaire et suffisante pour que dw/dz soit indépendante de n est que

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \alpha \right) \alpha = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \alpha$$

qui exige

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} - \nu \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y} = \mu \frac{\partial v}{\partial x}.$$

Ces conditions de monogénéité contiennent comme cas particuliers ces conditions de monogénéité pour les fonctions de variable complexe ordinaire, hyperbolique et dual.

En effet, quand

$$\nu=0, \mu=-1, \partial u/\partial x = \partial v/\partial y \text{ et } \partial v/\partial x = -\partial u/\partial y,$$

$$\nu=0, \mu=1, \partial u/\partial x = \partial v/\partial y \text{ et } \partial v/\partial x = \partial u/\partial y,$$

$$\nu=0, \mu=0, \partial u/\partial x = \partial v/\partial y \text{ et } \partial u/\partial y = 0.$$

En admettant l'existence des dérivées seconds de u et v , et si les théorèmes de permutabilité de la dérivation tiennent, on trouve les relations

$$\frac{\partial^2 u}{\partial y^2} - \mu \frac{\partial^2 u}{\partial x^2} - \nu \frac{\partial^2 u}{\partial x \partial y} = 0, \quad \frac{\partial^2 v}{\partial y^2} - \mu \frac{\partial^2 v}{\partial x^2} - \nu \frac{\partial^2 v}{\partial x \partial y} = 0,$$

équations qui contiennent comme cas particuliers celles de Laplace et celles déjà connues pour les variables complexes hyperboliques et duels.

Il en résulte, en outre, que la fonction holomorphe reste déterminée par ses valeurs de contour comme on démontre dans un travail publié.

Interprétation géométrique de le dérivé radial. L'expression de le dérivé radial

$$(3) \quad \left(\frac{dw}{dz} \right)_n = \frac{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \alpha \right) + n \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \alpha \right)}{1 + n\alpha} = X + Y\alpha,$$

où $\Delta y/\Delta x = n$, admet une interprétation géométrique simple et intéressante. En effet, de la (3) il résulte

$$(4) \quad \begin{aligned} X &= \frac{u_x + nu_y + mu_xv + m^2u_yv - v_xm\mu - v_y m^2\mu}{m\nu - m^2\mu + 1}, \\ Y &= \frac{v_x + mv_y - mu_x - m^2u_y\mu}{m\nu - m^2\mu + 1}. \end{aligned}$$

Si nous représentons les variables $z = x + y\alpha$, $w = u + v\alpha$ dans deux plans respectivement et si nous adoptons un troisième plan pour représenter

$$\left(\frac{dw}{dz} \right)_n = X + Y\alpha,$$

à chaque point z_0 de z correspond un point w_0 de w , tandis que pour le plan de dw/dz il résulte des infinis points correspondants aux infinies valeurs que peut prendre n .³

Nous allons montrer maintenant que ces points déterminent une conique.

En effet, les relations (4) peuvent s'écrire

$$X = \frac{\begin{vmatrix} u_x + nu_y & n\mu \\ v_x + mv_y & 1 + n\nu \end{vmatrix}}{\begin{vmatrix} 1 & n\mu \\ n & 1 + n\nu \end{vmatrix}}, \quad Y = \frac{\begin{vmatrix} 1 & u_x + nu_y \\ n & v_x + mv_y \end{vmatrix}}{\begin{vmatrix} 1 & n\mu \\ n & 1 + n\nu \end{vmatrix}},$$

d'où il résulte que X et Y sont les racines du système.

³ Kasner, *A complete characterization of the derivative of a polygenic function*, Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 172-177.

$$X + n\mu Y = u_x + nu_y, \quad nX^2 + (1 + n\nu)Y = v_x + nv_y.$$

Pour chaque valeur de n , la solution de ce système représente un point du plan et nous obtiendrons le lieu géométrique en faisant disparaître n entre les deux. Alors il résulte que

$$(5) \quad X^2 - \mu Y^2 + AY - BX + \nu XY + C = 0,$$

qui est la conique cherchée, avec

$$A = \frac{\partial y}{\partial y} - \nu \frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial x}; \quad B = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y};$$

$$C = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}.$$

On a pour

$\nu^2 + 4\mu > 0$, la (5) est type hyperbole,

$\nu^2 + 4\mu = 0$, la (5) est type parabole,

$\nu^2 + 4\mu < 0$, la (5) est type ellipse.

En particulier pour $\nu = 0$, $\mu = -1$, la (5) donne

$$(6) \quad (X - A_1)^2 + (Y - B_1)^2 = R_1^2$$

avec

$$A_1 = \frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right); \quad B_1 = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right);$$

$$R_1^2 = \frac{1}{4} \left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right]$$

qui est le cercle de Kasner.⁴ Pour $\nu = 0$, $\mu = 1$, la (5) donne

$$(7) \quad (X - A_2)^2 - (Y - B_2)^2 = R_2^2$$

avec

$$A_2 = \frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right); \quad B_2 = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right);$$

$$R_2^2 = \frac{1}{4} \left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 - \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right)^2 \right]$$

qui est la hyperbole dérivée. Pour $\nu = 0$, $\mu = 0$, la (5) donne

⁴ Kasner and De Cicco, *The derivative circular congruence-representation of a polygenic function*, American Journal of Mathematics, vol. 61 (1939), pp. 995-1003.

$$(8) \quad X^2 + \frac{\partial u}{\partial y} Y - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) X + \left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} \right) = 0$$

qui est l'équation de la parabole dérivée.

Considérations géométriques sur les coniques dérivées. On constate facilement que les points

$$\begin{aligned} M \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} \right), & \quad N \left(\frac{\partial u}{\partial x}, \frac{1}{\mu} \frac{\partial u}{\partial y} \right), \\ P \left(\frac{\partial v}{\partial y} - \nu \frac{\partial v}{\partial x}, \frac{\partial v}{\partial x} \right), & \quad Q \left(\frac{\partial v}{\partial y} - \frac{\nu}{\mu} \frac{\partial u}{\partial y}, \frac{1}{\mu} \frac{\partial u}{\partial y} \right), \end{aligned}$$

situés dans les sommets d'un trapèze rectangle appartient à la conique dérivée avec la notable propriété que le point M est indépendant du champ élu.

Quand le déterminant Δ du champ complexe varie avec continuité, la conique passe par déformation continue d'hyperbole et d'ellipse en parabole, avec la caractéristique que le point M demeure invariable.

La conique dérivée est générée par les faisceaux de droites

$$\begin{aligned} Y &= -\frac{1}{n\mu} X + \frac{1}{n\mu} \left(\frac{\partial u}{\partial x} + n \frac{\partial u}{\partial y} \right), \\ Y &= -\frac{n}{1+n\nu} X + \frac{1}{1+n\nu} \left(\frac{\partial v}{\partial x} + n \frac{\partial v}{\partial y} \right) \end{aligned}$$

qui passent par les points P et N .

De plus on prouve que la conique est projective avec le faisceau selon lequel on détermine les dérivées radiaux.

Il est notable d'observer qu'il existe une ressemblance entre la figure du voisinage et la conique dérivée, de telle façon que à voisinage elliptique, paire de droites parallèles et voisinage hyperbole, une ellipse, une parabole et une hyperbole correspondent respectivement comme conique dérivée.

D'autre part, les équations (6), (7) et (8), quand on remplit les conditions de monogénéité, se transforment en paires de droites (imaginaires dans le champ complexe elliptique), dont l'interception est le dérivé holomorphe dans le sens ordinaire.

A ce point de vue il existent infinis dérivés pour la fonction holomorphe, données par le point d'interception des droites (dérivé holomorphe dans le sens ordinaire), et les infinies valeurs qui correspondent aux directions isotropes.

Cette interprétation est en parfaite harmonie avec le concept de

point généralisé que nous venons d'exposer; de manière que nous pouvons dire encore que le dérivé holomorph d'une fonction de variable complexe binaire est un *point elliptique* pour la variable complexe ordinaire, un *point parabolique* pour le champ complexe parabolique et un *point hyperbolique* pour le champ complexe hyperbolique.

De la même façon la conique dérivée pour la fonction polygène est formée par points elliptiques, paraboliques et hyperboliques, selon le cas.

Quand on accomplit les conditions de monogénéité, la conique se déforme comme nous avons déjà dit, de manière que les points généralisés se transforment dans le point généralisé correspondant à le dérivé unique.

Remarque. Nous faisons noter que dans le champ complexe plus général de deux dimensions $a\alpha_1 + b\alpha_2$ telle que $\alpha_1^2 = \mu_1\alpha_1 + \nu_1\alpha_2$; $\alpha_2^2 = \mu_2\alpha_1 + \nu_2\alpha_2$; $\alpha_1\alpha_2 = \mu_3\alpha_1 + \nu_3\alpha_2$, formellement, il résulte pour les fonctions polygènes du variable de ce champ la conique suivante

$$\begin{aligned} & \begin{vmatrix} \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \end{vmatrix} X^2 + \begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix} Y^2 + \begin{vmatrix} \mu_1 & \nu_1 \\ \mu_3 & \nu_3 \end{vmatrix} XY \\ & - \left(\begin{vmatrix} \mu_1 & \nu_x \\ \mu_2 & \nu_y \end{vmatrix} - \begin{vmatrix} \nu_1 & u_x \\ \nu_2 & u_y \end{vmatrix} \right) X \\ & - \left(\begin{vmatrix} \mu_2 & \nu_x \\ \mu_3 & \nu_y \end{vmatrix} - \begin{vmatrix} \nu_2 & u_x \\ \nu_3 & u_y \end{vmatrix} \right) Y + \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = 0 \end{aligned}$$

qui contient toutes les antérieures comme des cas particuliers.

Je me limite à ces résultats fondamentaux des fonctions polygènes du nombre binaire, pour ne pas m'étendre trop sur cela, mais je vais indiquer seulement quelques résultats, sans démonstration pour ne pas entrer en détails.

Représentation conforme. Nous définissons la représentation conforme dans le champ binaire général comme étant une correspondance biunivoque qui conservent les arguments précédemment définis, c'est-à-dire, dans le sens de *conservation* d'aire. Nous démontrons que dans le champ complexe général on conserve la relation fondamentale entre l'holomorphisme et la représentation conforme, c'est-à-dire, *que toute fonction holomorphe conserve les arguments.*

Fonction linéaire (de Moebius). Nous faisons l'étude analogue au classique sur la fonction $w = (az+b)/(cz+d)$ démontrant que le résultat classique de la conservation de circonférences subsiste dans le champ général.

Or, nous démontrons que la fonction linéaire change la conique en conique (hyperboles dans le champ complexe hyperbolique; droites dans le champ parabolique et ellipses dans le champ elliptique) et réciproquement (données deux coniques il existe une fonction linéaire qui les fait correspondre)—à la condition que ces coniques soient homo-asymptotiques et des asymptotes parallèles aux droites isotropes.

Cette restriction apparente n'existe pas, parce que dans le champ classique toutes les circonférences sont homo-asymptotiques.

J'ai étudié également quelques autres sujets ayant rapport avec la fonction linéaire.

BUENOS AIRES, ARGENTINA

CREMONA INVOLUTIONS DETERMINED BY TWO LINE CONGRUENCES

EDWIN J. PURCELL

1. Introduction. Recently, in this Bulletin,¹ we discussed a multiple null-correspondence formed by two line congruences. The *first congruence* consisted in the lines intersecting a fixed twisted curve of order m and also its $(m-1)$ -secant d , and the *second congruence* consisted in the lines intersecting another twisted curve δ'_n , of order n , and its $(n-1)$ -secant d' .

When $m=2$, the first curve is a conic c_2 having one point on d ; a series of space Cremona involutions may now be defined as follows:

A generic point P determines a ray ρ_1 of the first congruence and a ray ρ' of the second congruence. In the null-plane of P , formed by ρ_1 and ρ' , lies another ray ρ_2 of the first congruence which intersects ρ' in P' , the correspondent of P in the involution.

2. The defining curves in general position. Let the equations of d be $x_1=0, x_2=0$; and those of d' be $x_3=0, x_4=0$. Let the parametric equations of c_2 and of δ'_n be, respectively,

$$\begin{aligned} x_1 &= \lambda^2, & x_1 &= f_n(s, t), \\ x_2 &= \lambda(\lambda - \mu), & x_2 &= F_n(s, t), \\ x_3 &= \lambda(\lambda - \mu), & x_3 &= \prod_{i=1}^{n-1} (t_i s - s_i t)(as + bt), \\ x_4 &= \mu^2, & x_4 &= \prod_{i=1}^{n-1} (t_i s - s_i t)(cs + dt), \end{aligned}$$

where s_i, t_i are the values of the parameters of δ'_n at the $n-1$ points on d' .

Then the equations of the involution are

$$\begin{aligned} x'_1 &= (x_2 - x_3)(x_1 F - x_2 f)^2, \\ x'_2 &= (x_1 F - x_2 f)L, \\ x'_3 &= x_3 \{ (x_2 - x_3)(x_1 F - x_2 f)(F - kHx_2) + (kHx_1 - f)L \}, \\ x'_4 &= x_4 \{ (x_2 - x_3)(x_1 F - x_2 f)(F - kHx_2) + (kHx_1 - f)L \}, \end{aligned}$$

where $f \equiv f_n(dx_3 - bx_4, ax_4 - cx_3)$, $k \equiv (ad - bc)$, $F \equiv F_n(dx_3 - bx_4,$

¹ E. J. Purcell, *A multiple null-correspondence and a space Cremona involution of order $2n-1$* , this Bulletin, vol. 46 (1940), pp. 339-344.

$$ax_4 - cx_2), H \equiv \prod_{i=1}^{n-1} \{ (s_i c + t_i d)x_3 - (s_i a + t_i b)x_4 \}, L \equiv (x_2 - x_3)(x_1 F - x_2 f) - (x_1 - x_3)(kHx_1x_3 - kHx_2x_3 + Fx_3 - Fx_1 - fx_3 + fx_4) - x_1x_4(F - kHx_2).$$

The order of the involution is $2n+3$, where n is any integer. The fundamental system and its principal images follow.

Line d is a simple F -line of simple contact. The fixed tangent plane to all the homaloids is $x_1=0$, which is the plane through d tangent to c_2 . d is of the first species and its P -surface is a ruled surface, $x_1F - x_2f = 0$, of order $n+1$ formed by the rays of the second congruence which intersect d . Each such ray goes over into its point of intersection with d . On this P -surface, d is a simple directrix line, δ'_n is a simple directrix curve, and d' is an n -fold directrix line.

Line d' is a $2n$ -fold F -line without contact. It is of the first species. Its P -surface is of order $2n+2$, whose equation is $(x_2 - x_3)(x_1F - x_2f)(F - kHx_1) + (kHx_1 - f)L = 0$. On this P -surface d is a simple line, d' is a $(2n-1)$ -fold line, c_2 is a simple conic, and δ'_n is a double curve; by their intersections with these curves, all the F -lines of second species have more than $2n+2$ points on the P -surface and therefore lie entirely on this surface.

Conic c_2 is a simple F -conic without contact. It is of the first species and has for P -surface the ruled surface of order $2n+2$ formed by the rays of the second congruence which intersect c_2 . Each such ray goes over into its point of intersection with c_2 . On this P -surface c_2 is a simple directrix curve, δ'_n is a double directrix curve, and d' is a $2n$ -fold directrix curve.

Curve δ'_n is a double F -curve without contact. It is of the first species and has for P -surface $(x_2 - x_3)(x_1F - x_2f)F - fL = 0$, which is of order $2n+2$. On this P -surface d is a simple line, d' is a $2n$ -fold line, c_2 is a simple conic, and δ'_n is a simple curve. Since $x_1=0$ is tangent to this surface along d , all the F -lines of second species lie on this P -surface.

The first is a $(1, 2)$ congruence and the second is a $(1, n)$ congruence. There are thus $2n+1$ common rays of both congruences. Each is a simple F -line of simple contact and is of the second species. The contact along these lines is torsal—that is, while all the homaloids have the same fixed tangent plane at a particular point on the line, the fixed tangent plane turns as this point moves along the line.

In the plane of c_2 are n rays of the second congruence. Each is a simple F -line without contact and is of the second species.

The fixed tangent plane, through d , to all the homaloids contains n rays of the second congruence. Each is a simple F -line without contact and is of the second species.

The locus of invariant points is a surface Σ of order $n+3$. Its equa-



tion is $x_1L - x_2(x_2 - x_3)(x_1F - x_2f) = 0$. On this surface d is a double line, d' is an n -fold line, c_2 is a simple conic, and δ'_n is a simple curve.

An alternate definition of the same involution employs this surface Σ . A generic point P determines a ray p' intersecting d' once and δ'_n once. Ray p' has with Σ two intersections, α and β , not on d' or δ'_n . We define P' , the correspondent of P in the involution, to be the harmonic conjugate of P with respect to α and β .

3. Lower order for d' in special position. Any plane through d' intersects δ'_n in one point Z not on d' .

If d' lies in the plane of c_2 , this plane will contain such a point Z . Any line in the plane of c_2 passing through Z is a ray of the second congruence. Each such line intersects d' once, δ'_n once, and c_2 twice. It has $2n+4$ intersections with every homaloid and therefore lies on every homaloid. The totality of these lines is the plane of c_2 which, being part of every homaloid, reduces the order of the involution by one.

The fundamental system is modified and the locus of invariant points is a surface of order $n+2$ on which d' is an $(n-1)$ -fold line, d is a double line, and δ'_n is a simple curve.

If the fixed tangent plane along d to all the homaloids contains d' , every line on it passing through Z will lie on every homaloid. This plane will factor from the equations of the involution and their order will be one lower than in the general case.

Here the locus of invariant points is a surface, Σ , of order $n+2$ on which d' is an $(n-1)$ -fold line, δ'_n is a simple curve, c_2 is a simple conic; but d does not lie on Σ .

4. The defining curves on one quadric surface. Let d , d' , c_2 , and δ'_n lie on the same quadric surface, d and d' being generators of the same system. Since c_2 intersects d once, it intersects d' once. δ'_n has $n-1$ points Δ'_i on d' and therefore $n-1$ points Δ_i on d . It follows that c_2 and δ'_n intersect in n points. Every generator of the other system intersects d , d' , c_2 , and δ'_n , each once and thus has $1+2n+1+2=2n+4$ intersections with every homaloid. Their locus is the quadric surface which is thus a part of every homaloid and factors from the equations of the involution, reducing the order by two. The involution in this case is of order $2n+1$.

If the quadric surface is $x_1x_3 - x_2x_4 = 0$; d is $x_1 = 0$, $x_2 = 0$; d' is $x_3 = 0$, $x_4 = 0$; and the parametric equations of c_2 and δ'_n are, respectively,

$$x_1 = \lambda\mu, \quad x_1 = \prod_{i=1}^{2n-2} (t_i s - s_i t) K(cs + dt),$$

$$\begin{aligned}
 x_2 &= \lambda^2, & x_2 &= \prod_n^{2n-2} (t_j s - s_j t) K(as + bt), \\
 x_3 &= \lambda \mu, & x_3 &= \prod_1^{n-1} (t_i s - s_i t)(as + bt), \\
 x_4 &= \mu^2, & x_4 &= \prod_1^{n-1} (t_i s - s_i t)(cs + dt),
 \end{aligned}$$

then the equations of the involution are

$$\begin{aligned}
 x_1' &= KW(Hx_1 - KWx_4)(x_1x_3 - x_2x_4), \\
 x_2' &= K^2W^2(x_1 - x_3)(x_1x_3 - x_2x_4), \\
 x_3' &= x_3(Hx_1 - KWx_4)(KWx_1 - Hx_3), \\
 x_4' &= x_4(Hx_1 - KWx_4)(KWx_1 - Hx_3),
 \end{aligned}$$

where K is a constant, H is as in §2, and

$$W = \prod_n^{2n-2} \{(s_j c + t_j d)x_3 - (s_j a + t_j b)x_4\}.$$

In this involution, d is a simple F -line without contact. It is of the first species and its P -surface is $x_1x_3 - x_2x_4 = 0$.

d' is a $(2n-1)$ -fold F -line without contact. It is of the first species and its P -surface is $KWx_1 - Hx_3 = 0$.

δ_n' is a simple F -curve of order n , of the first species, whose P -surface is $x_1 = 0$.

Δ_j , ($j = n, n+1, \dots, 2n-2$), the intersections of δ_n' and d , are isolated double F -points. Their P -surfaces are, respectively, $(s_j c + t_j d)x_3 - (s_j a + t_j b)x_4 = 0$.

$D_1, D_2, \dots, D_n$, the n intersections of c_2 and δ_n' , are isolated double F -points. The P -surface of each D_i is the plane of that D_i and d' .

Point $R(0100)$, the intersection of c_2 and d' , is a $2n$ -fold isolated F -point. Its P -surface is $Hx_1 - KWx_4 = 0$.

The F -curves of second species are as follows:

Lines RD_j , ($j = n, n+1, \dots, 2n-2$), are simple F -lines of simple contact.

Lines RD_i , ($i = 1, 2, \dots, n$), are simple F -lines without contact.

Through each D_i there passes a single line intersecting both d and d' . Each is a simple F -line without contact.

The plane of R and d , $x_1 = 0$, intersects the quadric surface in two lines, d and RS . $x_1 = 0$ intersects δ_n' in n points, $n-1$ of which are the Δ_j , ($j = n, n+1, \dots, 2n-2$), on d , and the remaining point is

on RS . Thus RS has $2n$ points of intersection with every homaloid at R , one at δ'_n , and one at d . Line RS is a simple F -line without contact.

Since $Q(1000)$ lies on d' , it is a $(2n-1)$ -fold point. But at Q , $n-1$ sheets of every homaloid have contact with the $n-1$ planes, $W=0$, through Q . The planes $W=0$ intersect d in Δ_j , ($j=n, n+1, \dots, 2n-2$). Each line $Q\Delta_j$ has $2n$ points of intersection with every homaloid at Q and two at Δ_j . Lines $Q\Delta_j$ are simple F -lines without contact.

The locus of invariant points is Σ , a surface whose equation is $Kx_1W(x_1-x_3)-x_3(Hx_1-KWx_4)=0$. Σ is of order $n+1$ and contains d' as $(n-1)$ -fold line without contact. d is a simple line on Σ , but neither δ'_n nor c_2 is on this surface. The alternate definition for this involution is: a generic point P determines a ray ρ' intersecting d' once and δ'_n once. Ray ρ' intersects Σ $n-1$ times on d' , and in two other points, α and β , not on d' . P' , the correspondent on P in the involution, is the harmonic conjugate of P with respect to α and β .

5. The second congruence consists in the bisecants of a space cubic curve. We now consider the involution of order 9 that results when the second congruence consists in the bisecants of a twisted cubic curve, g'_3 .

Let d be $x_1=0$, $x_3=0$. Let c_2 be $x_1=\lambda^2$, $x_2=\lambda\mu$, $x_3=\lambda^2$, $x_4=\mu^2$. Let the space cubic curve g'_3 be $x_1=\sigma^3$, $x_2=\tau^3$, $x_3=\sigma\tau^2$, $x_4=\sigma^2\tau$. Then the equations of the involution are

$$x'_1 = (x_1 - x_3)(C^2 - AB)^2,$$

$$x'_2 = (C^2 - AB)N,$$

$$x'_3 = (x_1 - x_3)A^2(C^2 - AB) - BCN,$$

$$x'_4 = B^2N - (x_1 - x_3)AC(C^2 - AB),$$

where $A \equiv x_3^2 - x_2x_4$, $B \equiv x_4^2 - x_1x_3$, $C \equiv x_1x_2 - x_3x_4$, and $N \equiv x_1^3x_3^2 - x_1^2x_3^3 + x_1^2x_2x_3x_4 + x_1x_3^2x_4^2 + x_3x_4^3 - x_1^2x_2^3 - 2x_2^2x_3^2x_4 + x_3^3x_4^2 - x_1x_2x_3^3 - x_2^2x_4^3 - x_1x_2x_3^2x_4 + 3x_1x_2^2x_3x_4 + x_1x_4^4 - 2x_1^2x_2x_4^2$.

The order of this involution is 9.

In this involution, d is a simple F -line of simple contact. It is of the first species and its P -surface, $C^2 - AB = 0$, is a ruled surface of order 4 consisting in the rays of the second congruence that intersect d . Each such ray goes over into its point of intersection with d .

c_2 is a simple F -conic of the first species whose P -surface is ruled and of order 8, being formed by the rays of the second congruence that intersect c_2 .

q'_3 is a fourfold F -cubic without contact. It is of the first species and its P -surface is of order 8.

Since the first congruence is of order 1 and class 2 and the second congruence is of order 1 and class 3, there are 7 common rays of both congruences. Each such ray intersects every homaloid 4 times at each of its two intersections with q'_3 , once on d , and once on c_1 . Thus the seven common rays of both congruences lie on every homaloid. They are simple F -lines of simple contact.

The fixed tangent plane, $x_1=0$, through d intersects q'_3 in three points in general. The three lines joining these points two and two are rays of the second congruence lying in $x_1=0$. Each such ray intersects every homaloid 4 times at each of its two points on q'_3 and twice on d . These three lines are simple F -lines without contact and are of the second species.

The plane of c_2 in general cuts q'_3 in three points. There are thus three rays of the second congruence which cut c_2 twice. They are simple F -lines without contact and are of the second species.

Σ , the locus of invariant points, is $x_1N - x_2(x_1 - x_3)(C^2 - AB) = 0$. This surface is of order 6 and contains q'_3 as a double cubic curve. d is a double line and c_2 is a simple conic on Σ .

Using this surface, we may define our involution: a generic point P determines a bisecant ρ' of the cubic curve q'_3 . Ray ρ' intersects Σ in two points, α, β , not on q'_3 . P' , the correspondent of P , is the harmonic conjugate of P with respect to α and β .

When d, c_2 , and q'_3 lie on the same quadric surface, with d intersecting q'_3 in one point, all the generators of the other system drop away and the order of the involution is lower by two. A discussion similar to that in §5 applies to this involution of order 7.

RATIONAL APPROXIMATIONS TO IRRATIONALS

ALEXANDER OPPENHEIM

It is well known that if p/q is a convergent to the irrational number x , then $|x - p/q| < 1/q^2$. The immediate converse is of course false but I have not seen in the literature¹ any statement of the converse which is given below.

THEOREM 1. *If p and q are coprime, $q > 0$, and if $|x - p/q| < 1/q^2$, then necessarily p/q is one of the three (irreducible) fractions*

$$p'/q', \quad (p' + p'')/(q' + q''), \quad (p' - p'')/(q' - q''),$$

where p''/q'' , p'/q' are two consecutive convergents to the irrational x . One at least of the two fractions $(p' + \epsilon p'')/(q' + \epsilon q'')$ where $\epsilon = \pm 1$ satisfies the inequality.

In other words if the inequality is satisfied, then

$$p/q = [a_1, a_2, \dots, a_{n-1}, a_n + c], \quad c = 0, \pm 1,$$

where $[a_1, a_2, \dots, a_r, \dots] = x$ is the infinite simple continued fraction for x , so that the a_i are integers, $a_i \geq 1$ ($i \geq 2$).

Suppose that $x - p/q = \epsilon\theta/q^2$, $0 < \theta < 1$, $\epsilon = \pm 1$. Let

$$p/q = [b_1, b_2, \dots, b_m], \quad p'/q' = [b_1, b_2, \dots, b_{m-1}],$$

where m (which we can choose to be odd or even) is taken so that $(-1)^{m-1} = \epsilon$. Defining y by the equation

$$x = [b_1, b_2, \dots, b_m, y] = (yp + p')/(yq + q'),$$

we obtain $\epsilon\theta = q^2(x - p/q) = (p'q - pq')q/(yq + q')$; so that, since $p'q - pq' = (-1)^{m-1} = \epsilon$, $y + q'/q = 1/\theta$.

Since $1/\theta > 1$ and $q'/q < 1$ it follows that $y > 0$.

If $y > 1$, then $y = [b_{m+1}, b_{m+2}, \dots]$ ($b_{m+1} \geq 1, \dots$), and so $x = [b_1, b_2, \dots, b_m, b_{m+1}, \dots]$, which, since the infinite simple continued fraction is unique, shows that $p/q = [b_1, \dots, b_m]$ is the m th convergent to x . If however $y < 1$, then $1/y = [c, b_{m+1}, b_{m+2}, \dots]$ with $c \geq 1$. But $q/q' = [b_m, b_{m-1}, \dots, b_2]$ and therefore one of c and b_m must be unity for, if not, then $1/y > 2$, $q/q' > 2$, $y + q'/q < 1 < 1/\theta$.

¹ Editor's note. In the meantime, R. M. Robinson has proved similar results in the Duke Mathematical Journal, vol. 7 (1940), pp. 354-359. Also the first part of Theorem 1 was observed by P. Fatou, Comptes Rendus de l'Académie des Sciences, Paris, vol. 139 (1904), pp. 1019-1021.

Hence $x = [b_1, \dots, b_{m-1}, b_m + c, b_{m+1}, \dots]$ and $b_i = a_i$ ($i \neq m$), $b_m + c = a_m$. Thus $p/q = [a_1, a_2, \dots, a_{m-1}, b_m]$ where $b_m = 1$ or $a_m - 1$. Consequently

$$p/q = (p_{m-1} + p_{m-2})/(q_{m-1} + q_{m-2}), \quad b_m = 1,$$

or

$$p/q = (p_m - p_{m-1})/(q_m - q_{m-1}), \quad b_m = a_m - 1.$$

The first part of the theorem is proved.

Now let $p/q = (p_n + \epsilon p_{n-1})/(q_n + \epsilon q_{n-1})$, $\epsilon = \pm 1$, p_n/q_n being the n th convergent to $x = [a_1, \dots, a_n, x'] = (x'p_n + p_{n-1})/(x'q_n + q_{n-1})$. Then

$$q^2 |x - p/q| = |x'\epsilon - 1| (q_n + \epsilon q_{n-1})/(x'q_n + q_{n-1}),$$

which, since $|\epsilon| = 1$, $x' > 1$, is less than unity if and only if

$$\epsilon(x' - q_n/q_{n-1}) < 2.$$

But this inequality is certainly satisfied when ϵ has the sign opposite to the sign of $x' - q_n/q_{n-1}$. The second part of the theorem follows.

Irreducible fractions p/q can be divided into three classes $[o/e]$, $[e/o]$, $[o/o]$ in which o and e denote odd and even integers respectively.

Since $p_n q_{n-1} - p_{n-1} q_n = \pm 1$ it is clear that consecutive convergents p_{n-1}/q_{n-1} , p_n/q_n belong to two different classes and hence that $(p_n + \epsilon p_{n-1})/(q_n + \epsilon q_{n-1})$ where $\epsilon = \pm 1$ must belong to the remaining class of irreducible fractions. It follows from Theorem 1 that for any irrational x infinitely many fractions of each class exist such that $|x - p/q| < 1/q^2$.

Theorem 1 in fact determines all such fractions.

This result is due to Scott² who used the geometric properties of elliptic modular transformations. Scott also showed that the result is the best possible: for a given class and a fixed k , $0 < k < 1$, irrationals exist, dense everywhere on the real axis, such that the inequality $|x - p/q| < k/q^2$ is satisfied by only a finite number of fractions in the given class.

To prove the last statement it will be enough to show that, if $x = [a_1, a_2, \dots, a_n, \dots]$ where the a_n are even integers not less than $2E+1$, where $E > 1$, then for every fraction of type $[o/o]$,

$$\theta = q^2 |x - p/q| > 1 - 1/E.$$

If $\theta > 1$, there is nothing to prove. If $\theta < 1$, it follows from our theo-

² W. T. Scott, this Bulletin, vol. 46 (1940), pp. 124-129.

rem that (p/q being irreducible)

$$p = p_n + \epsilon p_{n-1}, \quad q = q_n + \epsilon q_{n-1}, \quad \epsilon = \pm 1,$$

for the convergents to x are all $[e/o]$ or $[o/e]$. Write $X = [a_{n+1}, a_{n+2}, \dots]$, $Y = [a_n, a_{n-1}, \dots, a_2]$. Then if $n \geq 2$,

$$\begin{aligned} \theta &= \frac{(Y + \epsilon)(X - \epsilon)}{XY + 1} = 1 - \frac{2 - \epsilon(X - Y)}{XY + 1} > 1 - \frac{2 + X + Y}{XY + 1}, \\ XY + 1 - E(2 + X + Y) &= (X - E)(Y - E) - E^2 - 2E + 1 \\ &> (E + 1)^2 - E^2 - 2E + 1 > 0, \\ \theta &> 1 - 1/E. \end{aligned}$$

If $n=1$, then $p=p_1+1$, $q=q_1=1$, $\theta=1-[0, a_2, \dots] > 1-1/E$.

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MEASURABILITY AND DISTRIBUTIVITY IN THE THEORY OF LATTICES¹

M. F. SMILEY

Introduction. Garrett Birkhoff² derived the following self-dual symmetric condition that a metric lattice be distributive:

$$(1) \quad \begin{aligned} 2[\mu(a \cup b \cup c) - \mu(a \cap b \cap c)] &= \mu(a \cup b) - \mu(a \cap b) + \mu(a \cup c) \\ &\quad - \mu(a \cap c) + \mu(b \cup c) - \mu(b \cap c). \end{aligned}$$

In a previous note³ the author introduced and discussed a generalization of Carathéodory's notion of measurability⁴ with respect to an outer measure function μ which applies to arbitrary lattices L . The μ -measurable elements form a subset $L(\mu)$ consisting of those elements $a \in L$ which satisfy

$$(2) \quad \mu(a \cup b) + \mu(a \cap b) = \mu(a) + \mu(b)$$

for every $b \in L$. Closure properties of $L(\mu)$ were investigated. In par-

¹ Presented to the Society, January 1, 1941. The author wishes to express his gratitude to the referee for his valuable suggestions and comments.

² *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, p. 81. We shall adopt the notation and terminology of this work and shall indicate specific references to it by B.

³ *A note on measure functions in a lattice*, this Bulletin, vol. 46 (1940), pp. 239-241. We shall indicate references to this paper by M.

⁴ *Vorlesungen über Reelle Funktionen*, 2d edition, p. 246.

ticular, $L(\mu)$ was shown to be a sublattice of L when L is modular. It is our purpose here to quantify the relation (1) so as to provide a second (less wide) generalization of measurability. As before we restrict ourselves to modular lattices. A precise strengthening of (2) to yield (1) is first deduced. We then proceed to show that, at least for a positive functional μ , the same closure properties hold as before. We include an examination of the possibility of measurability of complements of measurable elements. Here the influence of distributivity is most apparent; since, under suitable hypotheses, the only measurable elements with "measurable" complements are the measurable neutral elements.

Our two important examples are metric lattices (B, p. 41) and the outer measure of Carathéodory. For the first of these we give, as an application of our method, new proofs of certain explicit and implicit results of V. Glivenko⁶ concerning relations between distributivity and metric betweenness.

1. Strongly measurable elements. Let us consider a modular lattice L with a least element $0 \in L$ and a functional $\mu(a)$ defined on L . Note that we do not require that μ be a modular functional. We define a subset $L_s(\mu)$ of strongly μ -measurable elements as those elements $a \in L$ which satisfy (1) for every⁶ $b \in L(\mu)$ and every $c \in L$.

LEMMA 1. *If an element $a \in L$ is strongly μ -measurable, then it is μ -measurable.*

PROOF. Take $b = 0 \in L(\mu)$ in (1) and the result is (2) with $b = c$.

A precise relation between strong measurability and distributivity is given in the following theorem.

THEOREM 1. *An element $a \in L$ is strongly μ -measurable if and only if it is μ -measurable and $\mu(a \cap (b \cup c)) = \mu((a \cap b) \cup (a \cap c))$ for every $b \in L(\mu)$ and every $c \in L$.*

PROOF. We base our proof on the one of Birkhoff (B, p. 81). First let $a \in L_s(\mu)$, and consider elements⁷ $b \in L(\mu)$, $c \in L$. By Lemma 1, $a \in L(\mu)$; and using (1) and (2) we obtain

⁶ *Contributions à l'étude des systèmes de choses normées*, American Journal of Mathematics, vol. 59 (1937), pp. 941-956, and *Géométrie des systèmes de choses normées*, ibid., vol. 58 (1936), pp. 799-828.

⁷ The quantification "for every $b, c \in L$ " would force $0 \notin L_s(\mu)$ or would make $L_s(\mu)$ empty except in the case $L = L(\mu)$. To see this take $a = 0$ in (1). Cf. the relations (a, b)D and (a)D of von Neumann's *Lectures on Continuous Geometry*, Princeton, 1935-1936, p. 38 ff.

$$\begin{aligned}
 (3) \quad & 2[\mu(a) + \mu(b \cup c) - \mu(a \cap (b \cup c)) - \mu(a \cap b \cap c)] \\
 & = \mu(a \cup b) - \mu(a \cap b) + \mu(a \cup c) \\
 & \quad - \mu(a \cap c) + \mu(b \cup c) - \mu(b \cap c).
 \end{aligned}$$

Since $b \in L(\mu)$, expansion of the right member of (3) yields

$$(4) \quad 2[-\mu(a \cap (b \cup c)) - \mu(a \cap b \cap c)] = -2\mu(a \cap b) - 2\mu(a \cap c).$$

It follows that

$$(5) \quad \mu(a \cap (b \cup c)) = \mu(a \cap b) + \mu(a \cap c) - \mu(a \cap b \cap c).$$

Now $L(\mu)$ is a sublattice of L (Theorem 1, M), hence $a \cap b \in L(\mu)$, and we have

$$(6) \quad \mu(a \cap (b \cup c)) = \mu((a \cap b) \cup (a \cap c)).$$

To establish the converse, let $a \in L(\mu)$ and consider elements $b \in L(\mu)$, $c \in L$. It is easy to retrace our steps through the equations (6)–(3) and thus obtain (1). This completes the proof.

Remark. By duality we may phrase the criterion of Theorem 1 as $\mu(a \cup (b \cap c)) = \mu((a \cup b) \cap (a \cup c))$ for every $b \in L(\mu)$ and every $c \in L$.

Remark. It is known that an element a of a modular lattice is neutral if and only if the correspondences $x \rightarrow x \cup a$ and $x \rightarrow x \cap a$ preserve the lattice operations (B, p. 59). The referee has pointed out the fact that $a \in L_s(\mu)$ if and only if these correspondences preserve measurability in the following sense. Define $\mu_a^+(x) \equiv \mu(a \cup x)$, $\mu_a^-(x) \equiv \mu(a \cap x)$. Then $a \in L_s(\mu)$ if and only if $a \in L(\mu)$ and $y \in L(\mu_a^+)$, $y \in L(\mu_a^-)$ for every $y \in L(\mu)$. To see this let $a \in L_s(\mu)$. By Lemma 1, $a \in L(\mu)$. Consider an element $y \in L(\mu)$. By Theorem 1 (M), $a \cup y \in L(\mu)$ and hence we have for every $x \in L$ that

$$\mu(a \cup x \cup y) + \mu((a \cup x) \cap (a \cup y)) = \mu(a \cup x) + \mu(a \cup y).$$

Using the preceding Remark, since $a \in L_s(\mu)$, $y \in L(\mu)$, we find that

$$(*) \quad \mu(a \cup (x \cup y)) + \mu(a \cup (x \cap y)) = \mu(a \cup x) + \mu(a \cap y).$$

This means that $y \in L(\mu_a^+)$. Dually $y \in L(\mu_a^-)$. Conversely, if $a \in L(\mu)$, $y \in L(\mu)$, and the equation (*) and its dual (which is equation (5)) are true, that is, if $y \in L(\mu_a^+)$ and $y \in L(\mu_a^-)$, then $a \in L_s(\mu)$ by the argument used in establishing the converse of Theorem 1.

2. **Examples.** (1) Let L_1 be the lattice of all subsets of a set S and μ_1 be an outer measure function (in the sense of Carathéodory⁷) de-

⁷ Op. cit., p. 238.

finer on L_1 . Here the auxiliary condition of Theorem 1 is automatically satisfied since L_1 is distributive. Hence $L_{1*}(\mu_1) = L_1(\mu_1)$ in this case. But we have shown (M, p. 239) that $L_1(\mu_1)$ consists of the subsets of S measurable (in Carathéodory's sense) with respect to μ_1 . Thus strong μ -measurability, though more restrictive than μ -measurability, still includes the ordinary idea of measurable set.

(2) Let L_2 be a metric lattice (in the sense of G. Birkhoff, B, p. 41) and μ_2 its modular functional. Then, since μ_2 is modular, $L_2(\mu_2) = L_2$; that is, all elements of L_2 are μ_2 -measurable. By Theorem 1, each neutral element of L_2 is strongly μ_2 -measurable. Conversely each strongly μ_2 -measurable element must be neutral. For the criterion of Theorem 1, the fact that $a \cap (b \cup c) \geq (a \cap b) \cup (a \cap c)$, and the sharply positive character of μ_2 imply that $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ for every $b, c \in L_2$ and dually.

Remark. For our second example the condition (1) holds as an inequality, the left-hand side being smaller.⁸ We see this easily by recalling that (6) holds as an inequality, and by proceeding as in the proof of the converse of Theorem 1. It is interesting to note also that if a lattice L has an O and an I the condition (1) cannot hold as an inequality (for every $a, c \in L$ and every $b \in L(\mu)$) unless $L = L(\mu)$, that is, unless μ is a modular functional. To see this take $b = O \in L(\mu)$ in (1) and obtain

$$\mu(a \cup c) + \mu(a \cap c) \leq \mu(a) + \mu(c).$$

Then take $b = I \in L(\mu)$ in (1) and the inequality sign is reversed. Thus (2) holds with b replaced by c for every $a, c \in L$; consequently μ is a modular functional.

3. Strongly measurable elements a sublattice. The closure of $L_*(\mu)$ with respect to the lattice operations $a \cap b$ and $a \cup b$ will be established for positive functionals μ .

THEOREM 2. *If μ is a positive functional, then $L_*(\mu)$ is a sublattice of L .*

PROOF. Let $a_1, a_2 \in L_*(\mu)$, and consider elements $b \in L(\mu)$, $c \in L$. It clearly suffices by duality, Lemma 1, Theorem 1 (M), and Theorem 1 to show that

$$\mu((a_1 \cap a_2) \cap (b \cup c)) = \mu((a_1 \cap a_2 \cap b) \cup (a_1 \cap a_2 \cap c)).$$

We always have $x \equiv (a_1 \cap a_2) \cap (b \cup c) \geq (a_1 \cap a_2 \cap b) \cup (a_1 \cap a_2 \cap c) \equiv y$, and hence $\mu(x) \geq \mu(y)$. The reverse inequality may be proved thus:

⁸ This was noted by the referee.

$$\begin{aligned}
 \mu(x) &= \mu(a_1) + \mu(a_2 \cap (b \cup c)) - \mu(a_1 \cup (a_2 \cap (b \cup c))) \\
 &= \mu(a_1) + \mu((a_2 \cap b) \cup (a_2 \cap c)) - \mu(a_1 \cup (a_2 \cap (b \cup c))) \\
 &\leq \mu(a_1) + \mu((a_2 \cap b) \cup (a_2 \cap c)) - \mu(a_1 \cup (a_2 \cap b) \cup (a_2 \cap c)) \\
 &= \mu(a_1 \cap ((a_2 \cap b) \cup (a_2 \cap c))) = \mu(y).
 \end{aligned}$$

The proof is complete. The author has been unable to decide the question of whether Theorem 2 remains true when the hypothesis that μ be positive is suppressed.

Let $N(L) \subset L$ denote the sublattice of neutral elements of L . Then we have $L(\mu) \supset L_s(\mu) \supset N(L) \cdot L(\mu)$. Examples will be given where each of these inequalities is strict. For the first, Example (2) of §2 will suffice; while a simple example, to be given in §5, settles the second.

4. Monotone sequences of elements of $L_s(\mu)$. The closure properties of $L_s(\mu)$ with respect to (o) -limits of monotone denumerable sequences can easily be related (as for $L(\mu)$) to continuity properties of the functional μ . We first restate the following definition (cf. M).

DEFINITION. *If for each sequence $a_i \uparrow a$ ($a_i \downarrow a$) of elements of $L(\mu)$ we have $\mu(a_i \cup b) \rightarrow \mu(a \cup b)$ and $\mu(a_i \cap b) \rightarrow \mu(a \cap b)$ for every $b \in L$, we say that (L, μ) satisfies B^+ (B^-).*

THEOREM 3. *If the functional μ is positive, then a sufficient condition that (o) -limits of increasing (decreasing) sequences of elements of $L_s(\mu)$ which belong to L also belong to $L_s(\mu)$ is that (L, μ) satisfy B^+ (B^-).*

PROOF. Let (L, μ) satisfy B^+ , and consider an increasing sequence of elements of $L_s(\mu)$: $a_i \uparrow a \in L$. By Lemma 1 and Theorem 2 (M) we know that $a \in L(\mu)$. For every $b \in L(\mu)$ and every $c \in L$ we have, by Theorem 1, that

$$\begin{aligned}
 \mu(a_i \cap (b \cup c)) &= \mu((a_i \cap b) \cup (a_i \cap c)) \\
 &= \mu(a_i \cap b) + \mu(a_i \cap c) - \mu(a_i \cap b \cap c).
 \end{aligned}$$

On taking the limit as $i \rightarrow \infty$ and using B^+ we get

$$\begin{aligned}
 \mu(a \cap (b \cup c)) &= \mu(a \cap b) + \mu(a \cap c) - \mu(a \cap b \cap c) \\
 &= \mu((a \cap b) \cup (a \cap c))
 \end{aligned}$$

for every $b \in L(\mu)$ and every $c \in L$. Thus $a \in L_s(\mu)$ by Theorem 1. The alternate reading is dual. The proof is complete. It is easy to formulate a sufficient condition on (L, μ) that the condition of Theorem 3 be necessary as well as sufficient.

5. Measurability of complements. In our Examples (1) and (2) of

§2 it is known that complements of strongly μ -measurable elements are unique and strongly μ -measurable.⁹ This is not true in general. In fact, complements of strongly μ -measurable elements may fail to be even μ -measurable. To see this, consider the simplest modular, non-distributive lattice L_0 of five elements O, a, b, c, I , with O and I least and greatest elements, respectively. Define $\mu(O)=0$, $\mu(I)=\frac{5}{8}$, $\mu(a)=\frac{3}{8}$, and $\mu(b)=\mu(c)=\frac{1}{8}$. It is easily seen that $a \in L_{0s}(\mu)$. However, it is obvious that neither b nor c (each a complement of a) belongs to $L_0(\mu)$. In this direction we may prove the following theorem.

THEOREM 4. *If μ is sharply positive, then a complement of a strongly μ -measurable element is μ -measurable if and only if it is unique, and then it is itself strongly μ -measurable.*

PROOF. Consider an element $a \in L_s(\mu)$ with a complement $x \in L(\mu)$. Suppose y is a second complement of a . By Theorem 1,

$$\mu(a \cap (x \cup y)) = \mu((a \cap x) \cup (a \cap y)) = \mu(O).$$

- Since μ is sharply positive $a \cap (x \cup y) = O$, and it follows that $x \cup y$ is a complement of a . But every complement z of a gives μ the same value; since, by (2),

$$(7) \quad \mu(O) + \mu(I) = \mu(a) + \mu(z).$$

Thus $\mu(x \cup y) = \mu(x) = \mu(y)$, and consequently, since μ is sharply positive, $x \cup y = x = y$. Hence x is unique. Conversely, let a' be the unique complement of a . Then by Theorem 4.5 (B), a and a' are neutral. To show that $a' \in L(\mu)$ (and hence that $a' \in L_s(\mu)$), consider an element $b \in L$. Since, by Lemma 1, $a \in L(\mu)$, we have

$$\begin{aligned} \mu(a) + \mu(a' \cap b) &= \mu(a \cup (a' \cap b)) + \mu(a \cap a' \cap b), \\ (8) \quad &= \mu((a \cup a') \cap (a \cup b)) + \mu(O), \\ &= \mu(a \cup b) + \mu(O). \end{aligned}$$

Dually,

$$(9) \quad \mu(a) + \mu(a' \cup b) = \mu(a \cap b) + \mu(I).$$

Adding equations (8) and (9), using (2) and (7), we find that

$$2\mu(a) + \mu(a' \cup b) + \mu(a' \cap b) = 2\mu(a) + \mu(a') + \mu(b).$$

This gives (2) with $a=a'$ at once. The proof is complete. We note

⁹ For example (1), see Carathéodory, op. cit., p. 248; while Birkhoff's Theorem 4.5 (B, p. 59) covers Example (2); or see the remark which closes this section.

that no use was made of the sharply positive character of the functional μ in establishing the converse of this theorem.

Remark. The converse of Theorem 4 shows that the complements of strongly μ -measurable elements will be strongly μ -measurable (and unique) whenever $L_*(\mu)$ consists entirely of neutral elements. This "explains" why this closure property holds in our Examples (1) and (2).¹⁰

6. **Relation to certain work of V. Glivenko.** Let us now consider in more detail our Example (2) of §2. V. Glivenko obtained, among other results, a characterization of elements of the center (the complemented neutral elements) of L_* in terms of metric betweenness properties.¹¹ Examination of his work will show that the neutral elements themselves may be similarly characterized in the following way.

LEMMA 2. *An element u of a metric lattice L is neutral if and only if $(a \cap b) \cup (u \cap (a \cup b))$ is between a and b for every $a, b \in L$.*

PROOF. We give a separate proof based on the condition (1). Consider an element $u \in L$ for which the condition holds. For every $a, b \in L$ we have, setting $x = (a \cap b) \cup (u \cap (a \cup b))$, $\delta(a, x) + \delta(x, b) = \delta(a, b)$, which becomes, in terms of μ ,

$$\begin{aligned} \mu(a \cup (u \cap (a \cup b))) - \mu((a \cap b) \cup (a \cap u)) + \mu(b \cup (u \cap (a \cup b))) \\ - \mu((a \cap b) \cup (b \cap u)) = \delta(a, b). \end{aligned}$$

Expansion of the left member gives

$$\begin{aligned} \mu(a) + \mu(b) + 2\mu(u \cap (a \cup b)) - 2\mu(a \cap u) - 2\mu(b \cap u) - 2\mu(a \cap b) \\ + 2\mu(a \cap b \cap u) = \delta(a, b). \end{aligned}$$

Further expansion shows that

$$\begin{aligned} \mu(a) + \mu(b) - 2\mu(u \cup a \cup b) + 2\mu(u) + 2\mu(a \cup b) - 2\mu(a \cap b) \\ - 2\mu(a \cap u) - 2\mu(b \cap u) + 2\mu(u \cap a \cap b) = \delta(a, b). \end{aligned}$$

Transposing, we have

$$\begin{aligned} \delta(a, b) + \mu(a) + \mu(u) - 2\mu(a \cap u) + \mu(b) + \mu(u) - 2\mu(b \cap u) \\ = 2\delta(u \cup a \cup b, u \cap a \cap b). \end{aligned}$$

¹⁰ It would be desirable, as the referee has pointed out, to construct an abstract theory which more completely unites our two examples. Our attempts in this direction have so far been unsatisfactory.

¹¹ Primus op. cit., p. 947. In a metric space, the point q is said to be between the point p and r in case $\delta(p, q) + \delta(p, r) = \delta(p, r)$. The reader is referred to L. M. Blumenthal's book *Distance Geometries* for a discussion of this relation.

Thus, finally,

$$\delta(a, b) + \delta(b, u) + \delta(u, a) = 2\delta(u \cup a \cup b, u \cap a \cap b),$$

which is merely the condition (1) written in terms of the metric δ . Therefore u is neutral. The converse may be proved by reversing the steps of our argument. The proof is complete.

We note that an element v is a complement of u if and only if both 0 and 1 are between u and v .

•For brevity's sake let us write (a, b, c) for the expression, "the element b is between the elements a and c ." Glivenko calls a betweenness relation *transitive*¹² if the relations (x, c, y) , (a, x, b) and (a, y, b) imply the relation (a, c, b) . He proves the facts of our final theorem.

THEOREM 5. *An element u of a metric lattice L is neutral if and only if either (i) $a \cap b \leq u \leq a \cup b$ implies that (a, u, b) for every $a, b \in L$, or (ii) metric betweenness is transitive in L .*

PROOF. Using the condition (1) we prove the first of these results in the following way. Suppose $a \cap b \leq u \leq a \cup b$, then

$$\begin{aligned}\delta(a, b) &= \mu(a \cup b) - \mu(a \cap b) = \mu(a \cup b \cup u) - \mu(a \cap b \cap u) \\ &= \delta(a \cup b \cup u, a \cap b \cap u).\end{aligned}$$

Hence, if (a, u, b) , then

$$2\delta(a \cup b \cup u, a \cap b \cap u) = 2\delta(a, b) = \delta(a, b) + \delta(a, u) + \delta(u, b);$$

which is (1). Conversely, if (1) is valid and $a \cap b \leq u \leq a \cup b$, then $\delta(a, b) = \delta(a, u) + \delta(u, b)$, that is; (a, u, b) . This proves the first result. The second follows easily from Lemma 2 and the first. For, let metric betweenness be transitive and consider elements $u, a, b \in L$. Take $x = a \cap b$, $y = a \cup b$, $c = (a \cap b) \cup (u \cap (a \cup b))$. We then have (x, u, y) since $x \leq u \leq y$; and by the modularity of μ we know that (a, x, b) and that (a, y, b) . Transitivity of metric betweenness implies that $(a \cap b) \cup (u \cap (a \cup b))$ is between a and b for every $a, b \in L$. It follows from Lemma 2 that u is neutral. We can now follow Glivenko in a proof of the converse. If (x, u, y) , (a, x, b) , and (a, y, b) , then $x \cap y \leq u \leq x \cup y$, $a \cap b \leq x \leq a \cup b$, $a \cap b \leq y \leq a \cup b$, and it is easily seen that $a \cap b \leq u \leq a \cup b$. If u is neutral we then obtain that (a, u, b) from the first result; and metric betweenness is transitive in L .

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¹² Secundus op. cit. For further transitivity of this nature and their application to a definition of betweenness in general lattices, see *Transitivity of betweenness* by Everett Pitcher and the author.

BICOMPACTNESS OF CARTESIAN PRODUCTS

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1. **Introduction.** Tychonoff [7] was the first to prove that the cartesian product of any number of bicomact spaces is bicomact. Of the other proofs¹ in the literature [2, 6] perhaps the simplest is that of Tukey, which involves the notion of an *ultraproduct*. In the present note a proof of a rather general form of this theorem is given, using only simple machinery. It is shown that the same method can be used to prove that the cartesian product of any number of absolutely closed Hausdorff spaces is an absolutely closed Hausdorff space.

2. **Definitions.** The spaces considered are those in which an operation of *closure* $\bar{A}$ of a set A is defined in terms of *neighborhoods* in the usual way, that is, x is a point of $\bar{A}$ if and only if every neighborhood of x contains a point of A . It follows that the closure operation is *monotone*; in other words, $\bar{A} \subset \bar{B}$ if $A \subset B$. Conversely, any closure operation which is monotone can be defined in terms of neighborhoods. No assumptions are made about the neighborhoods of a point, except that when they exist, they are sets of points.

The *cartesian product* P of a collection of such neighborhood spaces $\{B_k\}$ is a space whose points p are all selections $\{p_k\}$ containing just one point p_k from each of the spaces B_k . Neighborhoods are defined in P as follows. To any neighborhood N_k in B_k of a coordinate p_k of p , there corresponds in the product space P the neighborhood W_k of p consisting of all points q of P whose coordinate q_k is in N_k . The intersection of any finite collection $\{W_{k_r}\}$, $r=1, \dots, n$, of neighborhoods of p of this type, such that no two subscripts k_r are the same, is also defined to be a neighborhood of p . This is the usual definition of cartesian product due to Tychonoff [7]. Note that it is not true in general that the intersection of any finite number of neighborhoods of p is a neighborhood of p .

A system S of sets is said to have the *finite intersection property* if every finite number of sets of S has at least one common point. It can be shown by a familiar argument, using Zorn's lemma or transfinite induction [4, 6, 8], that any system S of subsets of a given set

¹ See also J. W. Alexander, *Ordered sets, complexes, and the problem of compactification*, Proceedings of the National Academy of Sciences, U.S.A., vol. 25 (1939), pp. 296-298, and E. Čech, *On bicomact spaces*, Annals of Mathematics, (2), vol. 38 (1937), p. 830.

with the finite intersection property is contained in a *maximal* system $\mathcal{M}$ with this property. This is also a consequence of the theorem of M. H. Stone that every ideal in a Boolean algebra is contained in a maximal ideal [4].

A space B is said to be *bicompact* if there is at least one point common to the closures of the sets of any system of sets with the finite intersection property (Tukey [6]). This is equivalent in T -spaces to the usual definition of bicomactness.

3. **Bicompactness of cartesian products.** We prove the following theorem.

THEOREM 1. *The cartesian product P of any collection of bicompact spaces $\{B_k\}$ is bicompact.*

PROOF. Let S be any system of sets of P with the finite intersection property, and let $\mathcal{M}$ be a maximal system with this property containing S . Define the *projection* $\mathcal{M}_k$ of the system $\mathcal{M}$ on the space B_k to be the system whose sets consist of the coordinates in B_k of points of a set of $\mathcal{M}$. The system $\mathcal{M}_k$ clearly has the finite intersection property, since if any sets of $\mathcal{M}$ have a common point, their projections on B_k have a common point also.

Since B_k is bicompact, there is a point p_k which is common to the closures of all the sets of $\mathcal{M}_k$. The points $\{p_k\}$ selected in this way, one from each space B_k , are the coordinates of a point p of P . We wish to show that p is common to the closures of all sets of S .

It follows from the way p_k was selected that any neighborhood N_k of p_k has a point in common with each set of the system $\mathcal{M}_k$. Consequently in the product space P , the neighborhood W_k of p which corresponds to N_k , has a point in common with every set of $\mathcal{M}$. Since $\mathcal{M}$ is maximal, W_k must belong to $\mathcal{M}$, and likewise every intersection of a finite number of such neighborhoods $\{W_{k_r}\}, r=1, \dots, n$, must belong to $\mathcal{M}$. For $\mathcal{M}$, being maximal, must contain all finite intersections of its sets, since otherwise these finite intersections could be added to $\mathcal{M}$. Hence every neighborhood of p belongs to $\mathcal{M}$, and therefore every neighborhood of p has a point in common with every set of $\mathcal{M}$. Consequently p is in the closure of every set of $\mathcal{M}$ and therefore of S , which was to be proved.

4. **Absolutely closed Hausdorff spaces.** A Hausdorff space H is said to be *absolutely closed* if every homeomorphic image of H which is a subset of a Hausdorff space K is a closed subset of K [1, 3, 5].

THEOREM 2. *A Hausdorff space H is absolutely closed if and only if*

there is a point common to the closures of the sets of any system of open sets of H with the finite intersection property.

PROOF. Suppose the system $\mathcal{S}$ of open sets of H has the finite intersection property, but that no point of H is common to the closures of all sets of $\mathcal{S}$. Then $\mathcal{S}$ is contained in a maximal system $\mathcal{M}$ of open sets with these two properties. Extend the space H to K by adding to it an ideal point x , whose neighborhoods are all sets obtained by adding x to each set of $\mathcal{M}$, while neighborhoods of points other than x remain the same as in H . Then K is a Hausdorff space, since the closures of neighborhoods of x have only the point x in common. Consequently H is not absolutely closed, since its image in K is not closed.

Conversely, suppose H is not absolutely closed, but is homeomorphic to a subset H^* of a Hausdorff space K , where H^* is not closed in K . Let x be a point of $K - H^*$ which is in the closure of H^* , and consider the space $H^* \cup \{x\}$. The system $\mathcal{M}$ of open sets of H^* obtained by deleting x from all its open neighborhoods, has the finite intersection property, since x is a point of the closure of H^* . There is no point common to the closures of all the sets of $\mathcal{M}$, since $H^* \cup \{x\}$ is a Hausdorff space. This completes the proof.

THEOREM 3. *The cartesian product P of any collection of absolutely closed Hausdorff spaces $\{H_k\}$ is an absolutely closed Hausdorff space.*

The proof is parallel to that of Theorem 1, with *sets* replaced by *open sets*. In defining the cartesian product P , only open neighborhoods are used, and $\mathcal{S}$, $\mathcal{M}$ are systems of open sets. It is merely necessary to verify that the projection on H_k of an open set of P is open.

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LINEAR FUNCTIONALS AND INTEGRALS IN ABSTRACT SPACES¹

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In his addendum to Saks, *Theory of the Integral*, Banach² considers a Lebesgue integral defined in a manner quite similar to that of Daniell³ and remarks that no use is made of a measure. It is, however, quite easy to show that Banach's and Daniell's integrals are expressible as Lebesgue integrals whose measure functions are regular outer measures in the sense of Carathéodory.

In the first two sections a linear, non-negative functional is considered. Upper and lower functionals are associated with this functional, and by means of them inner and outer measures are defined. It is shown that if the inner and outer measures of a set coincide, the set is measurable. To establish the converse a continuity assumption is made in §3, and a representation theorem in terms of the Lebesgue integral is obtained. It is shown in §4 that the theorem of Lebesgue for term-wise integration holds for semi-uniformly convergent $\mathfrak{R}$ -systems.

1. **Preliminary definitions.** It will be convenient to consider two linear classes of real-valued functions defined on a completely arbitrary range $\mathfrak{P}$. The first of these sets will be symbolized by $\mathfrak{X}$ and it will be supposed to contain the absolute value of every function in it. The second set $\mathfrak{Z}$ is made up of all functions $z(p)$ such that for some x in $\mathfrak{X}$, $|z| \leq x$. In accordance with Daniell's notation the symbols $x_1 \vee x_2$, $x_1 \wedge x_2$ represent the larger and smaller, respectively, of the functions x_1 , x_2 at each place p . Both these functions are in $\mathfrak{X}$ since $x_1 \vee x_2 = x_2 + (x_1 - x_2) \vee 0$, $x_1 \wedge x_2 = x_1 - (x_1 - x_2) \vee 0$ and since $x \vee 0 = (x + |x|)/2$.

Throughout this paper we shall be concerned with a linear functional I which is defined on the class $\mathfrak{X}$ and is non-negative in the sense that $x \geq 0$ implies $I(x) \geq 0$. Associated with this functional I there are two others I^* , I_* which will be called the upper and lower functionals and are defined by the equations

$$I^*(z) = \text{g.l.b.}_{x \leq z} I(x), \quad I_*(z) = \text{l.u.b.}_{x \geq z} I(x).$$

¹ Presented to the Society, April 13, 1940.

² Op. cit., p. 320 ff.

³ *A general form of integral*, Annals of Mathematics, vol. 19 (1918), pp. 279-294.

For functionals defined on other linear subspaces of $\mathfrak{B}$ similar upper and lower functionals may be defined, and they will have the same properties as I^* , I_* . It is easy to show that $I_*(z) = -I^*(-z)$; $I_*(z) \leq I^*(z)$; $I^*(z_1 + z_2) \leq I^*(z_1) + I^*(z_2)$; $I_*(z_1 + z_2) \geq I_*(z_1) + I_*(z_2)$; for $r \geq 0$ that $I_*(rz) = rI_*(z)$, $I^*(rz) = rI^*(z)$; and that I_* , I^* are non-negative.

LEMMA 1.1. For every z $I^*(|z|) - I_*(|z|) \leq I^*(z) - I_*(z)$.

To prove this result we notice that if $x_1 \geq z$, $x_2 \geq -z$, then $x_1 \vee x_2 \geq |z|$ and $x_1 \wedge x_2 \geq -|z|$ and $x_1 \vee x_2 + x_1 \wedge x_2 = x_1 + x_2$. We have then $I^*(|z|) - I_*(|z|) \leq I(x_1 \vee x_2) + I(x_1 \wedge x_2) = I(x_1) + I(x_2)$ for all x_1 , x_2 such that $-x_2 \leq z \leq x_1$, from which the conclusion follows at once.

2. **An extension of I.** In this section we obtain an extension of the range of definition of I and are thereby enabled to introduce inner and outer measures into the discussion. Let $\mathfrak{V}$ be the set of all z such that $I_*(z) = I^*(z)$.

THEOREM 1. The set $\mathfrak{V}$ is a linear subclass of $\mathfrak{B}$ containing $\mathfrak{K}$ and on which there is a linear, non-negative functional L coinciding on $\mathfrak{K}$ with I . The set $\mathfrak{V}$ is such that if y_1, y_2 are in it so are $y_1 \wedge y_2, y_1 \vee y_2$, and it is furthermore such that a function z is in $\mathfrak{V}$ if and only if the relation

$$L_*(z) \equiv \text{l.u.b.}_{z \geq y} L(y) = L^*(z) \equiv \text{g.l.b.}_{z \leq y} L(y)$$

is satisfied.

To establish the theorem let $L(y)$ be the common value of $I^*(y)$ and $I_*(y)$. It follows quite easily that L is linear and non-negative. To complete the proof let us first show that L^* , L_* coincide with I^* , I_* . It suffices evidently to show that $L^*(z) = I^*(z)$. By definition $L^*(z) \leq L(y) \leq I(x)$ for $z \leq y \leq x$ and hence $L^*(z) \leq I^*(z)$. It also follows from the definition of L^* that there is a sequence $y_n \geq z$ such that $L^*(z) > L(y_n) - 1/n = I^*(y_n) - 1/n \geq I^*(z) - 1/n$ for every integer n , which shows that $L^*(z) \geq I^*(z)$. It remains now only to show that $|y|$ is in $\mathfrak{V}$, and this follows readily from Lemma 1.1.

For convenience let us call a set E an M -set or say it is in $\mathfrak{M}$ in case its characteristic function is in $\mathfrak{V}$, and let us assume hereafter that the function $z(p) = 1$ is in $\mathfrak{V}$. We may then define the measure mE of an arbitrary set E as the lower bound of $L(y_F)$ for all M -sets F containing E , y_F being the characteristic function of F , and we define the lower measure m_*E in an analogous way.⁴

⁴ In his thesis, E. T. Welters has considered similar definitions of measures. See *Set Functions and Measurability Conditions*, University of Michigan doctoral dissertation, 1936.

THEOREM 2. *The functions m_* , m are monotone increasing set-functions such that $m_*E \leq mE$ and for every finite number of sets E_n*

$$m(\sum E_n) \leq \sum mE_n, \quad m_*(\sum E_n) \geq \sum m_*E_n.$$

A set E is an M -set if and only if $m_*E = mE$.

The first parts of the theorem are easily proved, and it is clear that if E is an M -set then $m_*E = mE$. Conversely if $m_*E = mE$, there is a sequence of M -sets $F_n \subset E$ such that $L^*(z_E) < L(y_{F_n}) + 1/n \leq L_*(z_E) + 1/n$, since $L^*(z_E) \leq mE$. Hence $L_*(z_E) = L^*(z_E)$.

THEOREM 3. *Every M -set is measurable in the usual sense.*

To prove this result we note first that the set $\mathfrak{M}$ is restrictedly additive. It is then clear that $mX \cdot E + mX \cdot CE \leq L(y_{F \cdot E} + y_{F \cdot CE}) = L(y_F)$ for every set X and every M -set $F \supset X$ and hence $mX \cdot E + mX \cdot CE \leq mX$, from which it follows that E is measurable.

3. **A continuity assumption.** To proceed further it is desirable to make a continuity assumption. We assume that for every monotone decreasing sequence of non-negative functions x_n in $\mathfrak{X}$, $I_*(\lim x_n) \geq \lim I(x_n)$. An obvious but important corollary of this hypothesis is that the function $(\lim x_n)$ is in the set $\mathfrak{Y}$, provided that the sequence x_n is as described in the preceding sentence.

LEMMA 3.1. *The limit of a monotone decreasing sequence of non-negative functions y_n in $\mathfrak{Y}$ is again in $\mathfrak{Y}$, and $L(\lim y_n) = \lim L(y_n)$.*

To prove this lemma we recall that there is a double sequence of non-negative functions x_{mn} in $\mathfrak{X}$ such that $x_{mn} \leq y_n$ and $I_*(y_n) < I(x_{mn}) + 1/m2^n$. There is no loss in generality if we assume that for each n the sequence x_{mn} is monotone increasing in m since $x_{1n} = x_{1n}$, $\bar{x}_{mn} = \bar{x}_{m-1,n} \vee x_{mn}$ has the desired monotone property and $I_*(y_n) < I(\bar{x}_{mn}) + 1/m2^n$ since $x_{mn} \leq \bar{x}_{mn}$. We show next that there is another sequence of non-negative functions ξ_{mn} in $\mathfrak{X}$ which is monotone decreasing in n , monotone increasing in m and for which $\xi_{mn} \leq x_{mn}$. This sequence may be defined as follows: $\xi_{m1} = x_{m1}$, $\xi_{mn} = \xi_{m,n-1} \wedge x_{mn}$. By a proof like that given by Daniell⁵ it then follows that $L(y_n) < I(\xi_{mn}) + 1/m$. With the help of the inequalities $\lim_n \xi_{mn} \leq \lim_n y_n \leq y_n$, it is seen that $L(\lim_n \xi_{mn}) \leq L_*(\lim y_n) \leq L^*(\lim y_n) \leq \lim L(y_n) \leq \lim_n I(\xi_{mn}) + 1/m \leq L(\lim_n \xi_{mn}) + 1/m$; whence $L_*(\lim y_n) = L^*(\lim y_n)$ since $L(\lim_n \xi_{mn})$ is monotone increasing and bounded.

⁵ Daniell, loc. cit., p. 283 ff.

LEMMA 3.2. *To every set E there corresponds an M -set $F \supset E$ such that $mE = mF$ and $m(F - E) = mE - m_*E$.*

According to the definition of m there is a sequence F_n of M -sets each containing E , such that $mE > L(y_{F_n}) - 1/n$. With the help of the continuity hypothesis made above it then follows easily that $F = \bigcap F_n$ is in $\mathfrak{M}$ and that $L(y_F) \geq mE \geq L(y_F) = mF$. To prove the second part of the lemma one needs only to keep in mind Theorem 3 and make a proof quite like that of Carathéodory.⁶

THEOREM 4. *The class $\mathfrak{M}$ is completely additive and contains every measurable set E . The set-function m is a regular outer measure of Carathéodory.*

It is quite easy to show that the intersection of a denumerable number of sets in $\mathfrak{M}$ is again in $\mathfrak{M}$, and this fact coupled with the restricted additivity of $\mathfrak{M}$ suffices to prove that $\mathfrak{M}$ is completely additive. By the lemma just proved there is a set F in $\mathfrak{M}$ that contains E and for which $mF = mE$. Since E is measurable, we have $mF = mE + m(F - E)$ and hence $m(F - E) = 0$, which implies that $m_*E = mE$.

Before proceeding further with our proof let us show that if F_n is a monotone increasing sequence of measurable sets, then $\lim mF_n = m(\lim F_n)$. To prove this statement let $F = \lim F_n$ and notice that the characteristic functions of the sets $F - F_n$ form a monotone decreasing sequence converging to zero. Hence $L(y_F) - L(y_{F_n}) = mF - mF_n$ converges to zero.

The regularity of m follows from Lemma 3.2 and Theorem 3. If $\{E_n\}$ is an arbitrary sequence of sets, there is another sequence of measurable hulls $F_n \supset E_n$. It is thus true that $m(\sum E_i) \leq m(\sum F_i) \leq \sum mF_i = \sum mE_i$ and hence m is an outer measure.

THEOREM 5. *Every y in $\mathfrak{Y}$ is integrable and*

$$(3.1) \quad L(y) = \int y dm.$$

Conversely every integrable function is in $\mathfrak{Y}$.

It follows at once from the definitions of m and the integral that (3.1) holds when y is a simple function, and from Lemma 3.1 that $\lim L(y_n) = L(\lim y_n)$ for monotone increasing sequences for which $(\lim y_n)$ is in $\mathfrak{Y}$. It evidently suffices to restrict our attention to non-negative functions y , recall that they are limits of monotone increas-

⁶ *Vorlesungen über reelle Funktionen*, p. 262, Theorem 5.

ing sequences of simple functions, and use Lebesgue's theorem on the integration of monotone sequences.

To prove the converse let z be an arbitrary integrable function. There is then a monotone decreasing sequence $\{y_n\}$ of functions in $\mathfrak{Y}$ converging to z for which the relation

$$(3.2) \quad \lim_n L(y_n) = L^*(z)$$

holds. With the help of Lebesgue's theorem on term-wise integration, the equation (3.1) and the relation (3.2) we have at once the equality of $L^*(z)$ and the integral of z . In a quite similar fashion it follows that $L_*(z)$ is equal to the integral of z and thus to $L^*(z)$, from which it follows that z is in $\mathfrak{Y}$.

To obtain the following corollary it is convenient to let $\mathfrak{U}$ be a linear space of real-valued functions u on $\mathfrak{P}$ containing $|u|$ and $u(\mathfrak{P}) = 1$.

COROLLARY 1. *Let I be a linear and non-negative functional defined on $\mathfrak{U}$ and let it be such that $I(u_n)$ converges to zero whenever $\{u_n\}$ is a monotone decreasing sequence with limit zero. There is then a regular outer measure of Carathéodory such that*

$$I(u) = \int u dm, \quad u \in \mathfrak{U}.$$

By a method quite like that of Daniell⁷ we may extend I to be defined over a space $\mathfrak{X}$ which has the properties described in the first paragraphs of §1 and §3. Our corollary then follows immediately from the theorem.

4. The theorem of Lebesgue. Let $\mathfrak{L}$ be an arbitrary class of elements l and let R be a transitive binary relation between the elements of $\mathfrak{L}$, having the so-called compositive property which guarantees the existence of an element in the R -relation to each of a pair of pre-assigned elements. An $\mathfrak{L}R$ -system of numbers is then defined as a real-valued function a_l defined on $\mathfrak{L}$.

Recalling Theorem 5, we see that $\mathfrak{Y}$ is exactly the space of all integrable functions. If y_l is an $\mathfrak{L}R$ -system of integrable functions converging almost everywhere to an integrable y , then without imposing more stringent convergence properties, we could not hope to prove that the equation

$$(4.1) \quad \lim_l \int y_l dm = \int y dm$$

⁷ Daniell, loc. cit., pp. 285-287.

is valid. Suppose, for the moment, that every finite set of points has measure zero and that the measure of $\mathfrak{P}$ is different from zero. If we choose l to be a finite subset of $\mathfrak{P}$ and $y_l(p)$ to be one for p in l and zero elsewhere, then y_l evidently converges to one, and yet the equation (4.1) does not hold.

The system y_l is said to converge semi-uniformly⁸ to y in case to each $\epsilon > 0$ and p there is an $l_{\epsilon, p}$ such that the inequality

$$(4.2) \quad |y_l(p) - y(p)| < \epsilon$$

is satisfied by all $l R l_{\epsilon, p}$ and in case for each fixed ϵ the set of elements $l_{\epsilon, p}$, as p varies, is at most denumerable.

THEOREM 6. *Let y_l be an $\mathfrak{R}$ -system of measurable functions such that for all l in the R -relation to some l_0 the inequality $|y_l(p)| \leq y_0(p)$ holds, where y_0 is integrable. Then if y_l converges almost everywhere in a semi-uniform way to a measurable function y , the relation (4.1) is valid.*

It follows from the definition of semi-uniform convergence that there is a monotone increasing sequence l_n and a system of integers $n_{\epsilon, p}$ such that the inequality (4.2) is satisfied for $\epsilon > 0$ and $l R l_n$ when $n = n_{\epsilon, p}$. It is then evident that if $l_{1n} R l_n$ ($n = 1, 2, \dots$) the sequence obtained from y_l by setting $l = l_{1n}$ will converge to y , and then by Lebesgue's theorem for sequences we have

$$(4.3) \quad \lim_n \int y_{l_{1n}} dm = \int y dm.$$

If our theorem were false, there would be a sequence $l_{1n} R l_n$ such that

$$\left| \int (y_{l_{1n}} - y) dm \right| \geq \epsilon,$$

which contradicts (4.3).

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⁸ E. H. Moore, *General Analysis*, Part II, p. 41 and p. 49.

THE ROOTS OF A POLYNOMIAL AND ITS DERIVATIVE¹

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Consider a polynomial $f(z)$ of degree n , with the roots $\alpha_1, \dots, \alpha_n$ and its first derivative $f'(z)$ with roots $\beta_1, \dots, \beta_{n-1}$. The classical problem, where certain of the roots α_i are given and conclusions are drawn concerning the location of the roots β_i , will be considered in the converse. By means of an identity between these two sets of roots, two theorems will be given which restrict the location of the roots of the polynomial $f(z)$ when certain of the roots of its derivative, $f'(z)$ are equally placed on the unit circle.

THEOREM 1. *If $\alpha_1, \dots, \alpha_n$, the n roots of the polynomial $f(z)$ of degree n , are different from the q distinct roots $\beta_1, \dots, \beta_q$, $q \leq n-1$, of the derived polynomial $f'(z)$, then*

$$(1) \quad \sum_{j=1}^n \frac{1}{(\beta_1 - \alpha_j)(\beta_2 - \alpha_j) \cdots (\beta_q - \alpha_j)} = 0.$$

The expansion of $1/(\beta_1 - x) \cdots (\beta_q - x)$ into partial fractions when summed as x ranges over the values $\alpha_1, \dots, \alpha_n$, yields

$$\sum_x \frac{1}{(\beta_1 - x) \cdots (\beta_q - x)} = \sum_x \frac{A_1}{(\beta_1 - x)} + \cdots + \sum_x \frac{A_q}{(\beta_q - x)},$$

where the coefficients $A_1, \dots, A_q$ are expressions of $\beta_1, \dots, \beta_q$, and thus may be taken outside of the summations. Then the theorem follows since the remaining sums are all identically zero by the well known expression

$$\frac{f'(\beta_i)}{f(\beta_i)} = \sum_{j=1}^n \frac{1}{(\beta_i - \alpha_j)} = 0, \quad i = 1, 2, \dots, q.$$

In a recent paper by M. Marden² an identity is given between q roots of $f'(z)$ and p roots of $f(z)$ of degree n , $n = p + q - 1$. This identity gave a clue to the relation (1). In fact (1) may be derived by means of a complete induction operating on the Marden identity.

In order to study the effect that (1) has on the distribution of the roots of the polynomial $f(z)$, we introduce the following notation. Let

¹ Taken from a dissertation presented for the degree of Doctor of Philosophy in Yale University.

² M. Marden, *Takeya's problem*, Transactions of this Society, vol. 45 (1939), pp. 355-368.

g_j be an arbitrary axis through β_j making an angle of ψ with the axis of reals. Let $L_j^{(k)}$ be the line through α_k and β_j of length $r_j^{(k)}$. Let $\theta_j^{(k)}$ be the angle formed by g_j and $L_j^{(k)}$ measured in the positive direction from $L_j^{(k)}$ to g_j . Finally set $\alpha_k - \beta_j = r_j^{(k)} e^{(\theta_j^{(k)} + \psi)i}$, $j=1, 2, \dots, q$; $k=1, 2, \dots, n$. With these notations the identity (1) may be stated in terms of its real and imaginary parts in the forms

$$(2) \quad \sum_{k=1}^n \frac{\cos(\theta_1^{(k)} + \dots + \theta_q^{(k)})}{r_1^{(k)} \dots r_q^{(k)}} = 0,$$

$$(3) \quad \sum_{k=1}^n \frac{\sin(\theta_1^{(k)} + \dots + \theta_q^{(k)})}{r_1^{(k)} \dots r_q^{(k)}} = 0.$$

Now in particular let us suppose that the q distinct roots of the polynomial $f'(z)$ are on the unit circle at the points $\theta = 2j\pi/q$, $j=1, 2, \dots, q$, respectively. Then if ψ is a right angle, $L_j^{(k)}$, for all j , will have the equation

$$\rho \sin \theta - \sin 2j\pi/q = \cot \theta_j^{(k)} (\rho \cos \theta - \cos 2j\pi/q).$$

Since the lines $L_j^{(k)}$, for all j , intersect at α_k , a root of $f(z)$, we determine the location of α_k so that (2) or (3) will be satisfied, by requiring the $\cos(\theta_1^{(k)} + \dots + \theta_q^{(k)}) = 0$ or the $\sin(\theta_1^{(k)} + \dots + \theta_q^{(k)}) = 0$. The curve D , where the $\cos(\theta_1^{(k)} + \dots + \theta_q^{(k)}) = 0$, may be determined by the substitution of $L_j^{(k)}$ in the expression for the $\cot(\theta_1^{(k)} + \dots + \theta_q^{(k)}) = 0$, and the curve C , where the $\sin(\theta_1^{(k)} + \dots + \theta_q^{(k)}) = 0$, may be determined by the substitution of $L_j^{(k)}$ in the expression $[\cot(\theta_1^{(k)} + \dots + \theta_q^{(k)})]^{-1} = 0$. We determine the curves D and C if q is even.

Using the following expansions of the cotangent function and dropping the superscript k , we have D and C from

$$(4) \quad \begin{aligned} 0 &= \cot(\theta_1 + \dots + \theta_q) \\ &= S_q^q \cot \theta_j - S_{q-2}^q \cot \theta_j + S_{q-4}^q \cot \theta_j + \dots + (-1)^{q/2} \end{aligned}$$

and from

$$(5) \quad \begin{aligned} 0 &= [\cot(\theta_1 + \dots + \theta_q)]^{-1} \\ &= S_{q-1}^q \cot \theta_j - S_{q-3}^q \cot \theta_j + S_{q-5}^q \cot \theta_j - \dots \\ &\quad + (-1)^{q/2+1} S_1^q \cot \theta_j, \end{aligned}$$

respectively, where

$$\cot \theta_j = \frac{\rho \sin \theta - \sin 2j\pi/q}{\rho \cos \theta - \cos 2j\pi/q}, \quad j = 1, 2, \dots, q,$$

and $S_j^q \cot \theta_j$ means the sum of all different products that can be formed using the q functions $\cot \theta_1, \dots, \cot \theta_q$, r at a time. We also note that the $\cot (\theta_1 + \dots + \theta_q)$ equals the right side of equation (4) divided by the right side of equation (5). Thence the expressions for D and C will follow from the two lemmas.

LEMMA 1. If $F(\rho, \theta) = \rho^q \cos q\theta + \sum_{k=1}^{q-1} a_k \rho^k \cos k\theta + b_k \rho^k \sin k\theta = K$, and if $F(\rho, \theta)$ is identically equal to $F(\rho, \theta')$ where $\theta' = 2\pi/q + \theta$, then $a_k = b_k = 0$, $k = 1, 2, \dots, q-1$.

We compare the terms of like powers of ρ —

$$\rho^q: \quad \cos q\theta' = \cos q(2\pi/q + \theta) = \cos (2\pi + q\theta) = \cos q\theta,$$

$$\rho^k, 0 < k < q: \quad a_k \cos k\theta' + b_k \sin k\theta' = a_k \cos k\theta + b_k \sin k\theta,$$

or by expressing θ' in terms of θ and observing that the relation must hold for all values of θ , we have

$$\begin{aligned} a_k(\cos 2\pi k/q - 1) + b_k(\sin 2\pi k/q) &= 0, \\ -a_k(\sin 2\pi k/q) + b_k(\cos 2\pi k/q - 1) &= 0. \end{aligned}$$

For a solution of a_k and b_k which is not zero to exist, we get from the determinant of the coefficients that the $\cos 2\pi k/q = 1$ or $2\pi k/q = 2\pi t$, $t = 0, 1, \dots$. Hence $k/q = t$, which is impossible. Therefore the only solution is $a_k = b_k = 0$, for $k = 1, 2, \dots, q-1$.

LEMMA 2. If $F(\rho, \theta) = \rho^q \sin q\theta + \sum_{k=1}^{q-1} a_k \rho^k \cos k\theta + b_k \rho^k \sin k\theta = L$ and $F(\rho, \theta)$ is identically equal to $F(\rho, \theta')$ where $\theta' = 2\pi/q + \theta$, then $a_k = b_k = 0$, $k = 1, 2, \dots, q-1$.

The method of proof is obvious from Lemma 1. Since (4) satisfies the hypothesis of Lemma 1, we have D as $\rho^q \cos q\theta = K$. We also observe that the point $\rho = 1$, $\theta = 2\pi/q$ is on the curve D , from which we see that $K = 1$. Similarly (5) satisfies the hypothesis of Lemma 2 and thus C is $\rho^q \sin q\theta = L$. Also $\rho = 1$, $\theta = 2\pi/q$ is a point on C from which follows $L = 0$. Combining the above facts the following theorems may be stated.

THEOREM 2. If q distinct roots of the derivative of the polynomial $f(z)$, of degree n , are different from the roots of $f(z)$ and are located so that they divide the unit circle into q equal arcs, then either all of the roots of $f(z)$ are on the curve $\rho^q \cos q\theta = 1$, or there are roots in the interior as well as the exterior of the curve.

By the interior is meant all points of the plane which are not on the curve but which are separated from the point $\rho=0$ by the curve. By the exterior is meant all other points of the plane not on the curve nor in the interior of the curve.

THEOREM 3. *If, with the hypothesis of Theorem 2, the $2q$ radial lines through $\rho=0$ and the roots of $f'(z)$ and midway between the roots are drawn so that the plane is divided into $2q$ sectors, numbered from 1 to $2q$, then either all the roots of $f(z)$ must lie on the radial lines or there must be roots in an odd as well as in an even numbered sector.*

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ON THE REPRESENTATIONS, $N_7(m^2)^1$

C. D. OLDS

1. Introduction. Write $N_r(n)$ for the number of representations of the positive integer n as the sum of r squares, and write $N_r(n, k)$ for the number of representations of n as the sum of r squares in which the first k squares in each representation are odd with positive roots, while the remaining $r-k$ squares are even with roots positive, negative, or zero. In a previous paper the author [5]² gave an arithmetical derivation of the formula for $N_8(n^2)$. The method used to prove this result was based upon that employed by Hurwitz [2] in his discussion of the analogous formula for $N_8(n^2)$.

In 1930, G. Pall [6] gave an analytical derivation of the formula for $N_7(cn^2)$, c an integer. His formula shows, in particular, that if $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where $p_1, p_2, \dots, p_s$ are distinct odd primes, then

$$(1) \quad N_7(m^2) = 14 \prod_{r=1}^s [\rho_r, \alpha_r],$$

where

$$[\rho_r, \alpha_r] = \sigma_8(\rho_r^{\alpha_r}) - (-1)^{(p_r-1)/2} \rho_r^2 \sigma_8(\rho_r^{\alpha_r-1}).$$

We define the arithmetical function $\sigma_k(n)$, which occurs here, and the function $\rho_k(n)$, which occurs later, by the sums

¹ This is the second part of a paper presented to the Society, April 6, 1940, under the title *On the number of representations of the square of an integer as the sum of an odd number of squares*. The author wishes to thank Professor J. V. Uspensky for help in preparing this paper.

² The numbers in brackets refer to the bibliography.

$$\sigma_k(n) = \sum_{d|n} d^k, \quad \rho_k(n) = \sum_{n=d\delta} (-1)^{(\delta-1)/2} d^k, \quad \delta \text{ odd},$$

where in the first sum d ranges over all divisors of n : $1 \leq d \leq n$, and in the second d, δ range over all integral solutions of the equation $n = d\delta$, where δ is odd. It is the purpose of this paper to show how formula (1) may be derived arithmetically. We shall use the method of Hurwitz [2].

2. Preliminary formulas. In the proof of (1) which follows, we use certain well known results concerning $N_r(n)$, when r is even. These formulas, which are not difficult to prove arithmetically,³ are

$$(2) \quad N_6(m) = 12\rho_2(m), \quad m \equiv 1 \pmod{4};$$

$$(3) \quad N_6(2m, 2) = \rho_2(m), \quad N_6(2^{\alpha+2}m, 4) = 2^{2\alpha}\rho_2(m) = \chi(2^\alpha m), \quad \alpha \geq 0, m \text{ odd};$$

$$(4) \quad N_{12}(4m, 4) + 16N_{12}(4m, 8) = \sigma_6(m), \quad m \text{ odd}.$$

3. Three lemmas. The first lemma we need was proved by the author in [5].

LEMMA 1. *Let $f(n)$ be an arbitrary arithmetical function, not equal to zero for all n , and such that $f(nn') = f(n)f(n')$ for any two integers n and n' . If $F(n) = \sum f(d)$, where d ranges over all divisors of n , then*

$$(5) \quad F(nn') = \sum_{(a)} \mu(d)f(d)F(n/d)F(n'/d),$$

where (a) indicates that we take $d = 1, 2, 3, \dots$, with the convention that $F(x) = 0$ if x is not an integer. Here $f(1) = F(1) = 1$ and $\mu(d)$ denotes the Möbius function.

Now let $P = N_7(m^2)$, m an odd integer, and let P_1 denote the number of solutions of the equation

$$m^2 = x^2 + y^2 + z^2 + \bar{x}^2 + \bar{y}^2 + 4w^2,$$

and P_2 the number of solutions of the equation

$$m^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 + \bar{\bar{x}}^2 + \bar{\bar{y}}^2 + 4v^2 + 4w^2,$$

where the barred squares have odd roots and occupy the first five places in each representation. Then it may be proved without difficulty that

$$P = \frac{7}{6}(P_1 + 12P_2),$$

³ Formula (2) may be found, for example, in J. Liouville [3], formulas (3) and (4) in Liouville [4]. For the method used to prove such formulas arithmetically, see T. Pepin [7] or P. Bachmann [1].

and thus our problem is reduced to the determination of convenient expressions for P_1 and P_2 .

We first discuss P_1 . To this end, notice that

$$P_1 = \sum_{(b)} N_6(m^2 - 4w^2),$$

where (b) indicates that $w = 0, \pm 1, \pm 2, \dots; 4w^2 < m^2$. On the other hand, (2) shows us that

$$N_6(m^2 - 4w^2) = 12\rho_2(m^2 - 4w^2),$$

so that

$$\begin{aligned} P_1 &= 12 \sum_{(b)} \rho_2(m^2 - 4w^2) = 12 \sum_{(b)} \rho_2((m - 2w)(m + 2w)) \\ &= 12 \sum_{(c)} \rho_2(m'm''), \end{aligned}$$

where the last sum ranges over all positive odd integers m', m'' satisfying the equation (c): $2m = m' + m''$.

To evaluate $\rho_2(m'm'')$ use Lemma 1. Define

$$\begin{aligned} f(n) &= 0, \quad \text{whenever } n \text{ is even,} \\ f(n) &= (-1)^{(n-1)/2} n^2, \quad \text{whenever } n \text{ is odd;} \end{aligned}$$

then $f(nn') = f(n)f(n')$ for any two integers n, n' as required, and for odd $n (=m)$,

$$\begin{aligned} F(m) &= \sum_{d|m} f(d) = \sum_{d|m} (-1)^{(d-1)/2} d^2 = (-1)^{(m-1)/2} \sum_{\substack{d|m \\ d \text{ odd}}} (-1)^{(d-1)/2} d^2 \\ &= (-1)^{(m-1)/2} \rho_2(m), \quad \delta \text{ odd.} \end{aligned}$$

Applying (5) we obtain

$$(6) \quad \rho_2(m'm'') = \sum_{(a)} \mu(d) (-1)^{(d-1)/2} d^2 \rho_2(m'/d) \rho_2(m''/d),$$

provided we adopt the convention that $\rho_2(x) = 0$ if x is not an integer. Hence

$$\begin{aligned} P_1 &= 12 \sum_{(c)} \rho_2(m'm'') = 12 \sum_{(c)} \sum_{(a)} \mu(d) (-1)^{(d-1)/2} d^2 \rho_2(m'/d) \rho_2(m''/d) \\ &= 12 \sum_{d=1,3,5,\dots} \mu(d) (-1)^{(d-1)/2} d^2 \sum_{(c)} \rho_2(m'/d) \rho_2(m''/d). \end{aligned}$$

Finally, with the aid of (3), we find

$$\sum_{(c)} \rho_2\left(\frac{m'}{d}\right) \rho_2\left(\frac{m''}{d}\right) = \sum_{(c)} N_6\left(\frac{2m'}{d}, 2\right) N_6\left(\frac{2m''}{d}, 2\right) = N_{12}\left(\frac{4m}{d}, 4\right).$$

Thus we have established

LEMMA 2. $P_1 = 12 \sum_{d|m} \mu(d) (-1)^{(d-1)/2} d^2 N_{12}(4m/d, 4)$.

We evaluate P_2 in a similar manner. Since

$$P_2 = 2^4 \sum_{(d)} N_6(m^2 - x^2, 4),$$

where the extent of the summation is indicated by (d) : $x = \pm 1, \pm 3, \dots$; $x^2 < m^2$; we find, using (3), that

$$\begin{aligned} P_2 &= 16 \sum_{(d)} \chi\left(\frac{m^2 - x^2}{4}\right) = 16 \sum_{(d)} \chi\left(\frac{m+x}{2} \cdot \frac{m-x}{2}\right) \\ &= 16 \sum_{(e)} \chi(n'n''), \end{aligned}$$

where n', n'' are any two positive integers satisfying the equation (e): $m = n' + n''$. Setting

$$n' = 2^\alpha m', \quad n'' = 2^\beta m'',$$

where m', m'' are odd, and using (6), we deduce that

$$\begin{aligned} \chi(n'n'') &= 2^{2(\alpha+\beta)} \rho_2(m'm'') \\ &= \sum_{(a)} \mu(d) (-1)^{(d-1)/2} d^2 2^{2\alpha} \rho_2\left(\frac{m'}{d}\right) 2^{2\beta} \rho_2\left(\frac{m''}{d}\right) \\ &= \sum_{(a)} \mu(d) (-1)^{(d-1)/2} d^2 \chi\left(\frac{n'}{d}\right) \chi\left(\frac{n''}{d}\right). \end{aligned}$$

As a result

$$P_2 = 16 \sum_{(e)} \chi(n'n'') = 16 \sum_{d=1,3,5,\dots} \mu(d) (-1)^{(d-1)/2} d^2 \sum_{(e)} \chi\left(\frac{n'}{d}\right) \chi\left(\frac{n''}{d}\right).$$

Now by (3)

$$\sum_{(e)} \chi\left(\frac{n'}{d}\right) \chi\left(\frac{n''}{d}\right) = \sum_{(e)} N_6\left(\frac{4n'}{d}, 4\right) N_6\left(\frac{4n''}{d}, 4\right) = N_{12}\left(\frac{4m}{d}, 8\right).$$

We have proved

LEMMA 3. $P_2 = 16 \sum_{d|m} \mu(d) (-1)^{(d-1)/2} d^2 N_{12}(4m/d, 8)$.

4. Derivation of the formula for $N_7(m^2)$. Since we know already $P = \frac{1}{3}(P_1 + 12P_2)$, we find, using Lemma 2, Lemma 3, and finally (4), that

$$\begin{aligned}
P &= \frac{7}{6} \left[12 \sum_{d|m} \mu(d) (-1)^{(d-1)/2} d^2 N_{12} \left(\frac{4m}{d}, 4 \right) \right. \\
&\quad \left. + 12 \cdot 16 \sum_{d|m} \mu(d) (-1)^{(d-1)/2} d^2 N_{12} \left(\frac{4m}{d}, 8 \right) \right] \\
&= 14 \sum_{d|m} \mu(d) (-1)^{(d-1)/2} d^2 \left[N_{12} \left(\frac{4m}{d}, 4 \right) + 16 N_{12} \left(\frac{4m}{d}, 8 \right) \right] \\
&= 14 \sum_{d|m} \mu(d) (-1)^{(d-1)/2} d^2 \sigma_5 \left(\frac{m}{d} \right) \\
&= 14 \prod_{r=1}^s [p_r, \alpha_r],
\end{aligned}$$

where

$$[p_r, \alpha_r] = \sigma_5(p_r^{\alpha_r}) - (-1)^{(p_r-1)/2} p_r^2 \sigma_5(p_r^{\alpha_r-1}).$$

This is the result we set out to prove.

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A NOTE ON THE SPECIAL LINEAR HOMOGENEOUS GROUP $SLH(2, p^n)$

F. A. LEWIS

1. **Introduction.** The following theorem is due to E. H. Moore.

The special linear homogeneous group $SLH(2, p^n)$ of binary linear substitutions of determinant unity in the $GF[p^n]$ is simply isomorphic with the abstract group L generated by the operators T and S_λ , where λ runs through the series of p^n marks of the field, subject to the generational relations

$$(a) S_0 = I, S_\lambda S_\mu = S_{\lambda+\mu} \quad (\lambda, \mu \text{ any marks}),$$

$$(b) T^4 = I, S_\lambda T^2 = T^2 S_\lambda,$$

$$(c) S_\lambda T S_\mu T S_{(1-\lambda)/(1-\lambda\mu)} T S_{1-\lambda\mu} T S_{(1-\mu)/(1-\lambda\mu)} T = I \quad (\lambda, \mu \text{ any marks, } \lambda\mu \neq 1).$$

For $\lambda = 1, \mu \neq 1$, (c) gives

$$(d) (S_1 T^2)^2 = I.$$

Other relations employed by Dickson¹ in a proof of this theorem are

$$(e) T S_\alpha T S_{2\alpha-1} T S_\alpha T S_{2\alpha-1} T^2 = I \quad (\alpha \neq 0),$$

$$(f) T S_\alpha T S_{\alpha-1} T S_\rho = S_{\alpha-2\rho} T S_\alpha T S_{\alpha-1} T \quad (\rho \text{ any mark}).$$

It is the purpose of this paper to prove that (a), (b), (d), and (e) define an abstract group simply isomorphic with $SLH(2, p^n)$ when $p > 2$. If $p = 2$, relation (e) reduces to an identity and must be replaced by (f).

2. **Preliminary relations.** We first prove that (f) is a consequence of (a), (b), (d), and (e) when $p > 2$, so that in what follows we may use (f) for any p . We write (e) in the form

$$(e') T S_\alpha T = S_{-2\alpha-1} T S_{-\alpha} T S_{-2\alpha-1} T^2$$

and make an even number of applications of this formula to the right member of (f) as follows:

$$\begin{aligned} S_{\alpha-2\rho} T S_\alpha T \cdot S_{\alpha-1} T &= S_{\alpha-2\rho-2\alpha-1} T S_{-\alpha} \cdot T S_{-\alpha-1} T \cdot T^2 \\ &= S_{\alpha-2\rho-2\alpha-1} \cdot T S_\alpha T \cdot S_{\alpha-1} T S_{2\alpha} = S_{\alpha-2\rho-4\alpha-1} T S_{-\alpha} \cdot T S_{-\alpha-1} T \cdot S_{2\alpha} T^2 \\ &= S_{\alpha-2\rho-4\alpha-1} \cdot T S_\alpha T \cdot S_{\alpha-1} T S_{4\alpha} = S_{\alpha-2\rho-6\alpha-1} T S_{-\alpha} \cdot T S_{-\alpha-1} T \cdot S_{4\alpha} T^2 \\ &= S_{\alpha-2\rho-6\alpha-1} \cdot T S_\alpha T \cdot S_{\alpha-1} T S_{6\alpha} = \dots = S_{\alpha-2\rho-2m\alpha-1} \cdot T S_\alpha T \cdot S_{\alpha-1} T S_{2m\alpha}. \end{aligned}$$

Relation (f) is established by taking $m = \rho/2\alpha$. It will be convenient to write (f) in the equivalent form

$$(f') S_\rho T S_\alpha T S_{\alpha-1} T = T S_\alpha T S_{\alpha-1} T S_{\rho\alpha^2}.$$

¹ *Linear Groups*, Leipzig, 1901. The notation is that employed by Dickson.

Now let e be a primitive root of the field and define

$$(1) \quad R = T^2 S_e T^2 S_{e^{-1}} T^2 S_e.$$

Since

$$(2) \quad R^k = T^2 S_e T^2 S_{e^{-k}} T^2 S_e$$

is true by definition when $k=1$, by induction (2) holds for any k if

$$T^2 S_e T^2 S_{e^{-k}} T^2 S_e T^2 S_e T^2 S_{e^{-1}} T^2 S_e = T^2 S_{e^{k+1}} T^2 S_{e^{-k-1}} T^2 S_{e^{k+1}}$$

or

$$T^2 S_e T^2 S_{e^{-k}} T^2 S_e T^2 S_e T^2 S_{e^{-1}} T^2 S_e S_{e^{-k+1}} T S_{e^{-k-1}} T S_{e^{-k+1}} T = I.$$

Upon making obvious reductions this last relation becomes

$$S_{e^{-k+1}} T S_{e^{-k}} T S_e \cdot T S_e T S_{e^{-1}} T \cdot S_{e^{-k+1}} T S_{e^{-k-1}} T = T^2.$$

If we apply (f) as indicated, this becomes

$$S_{e^{-k+1}} T S_{e^{-k}} T S_{e^{-k-1}+e^{-1}} T S_e T S_{e^{-1}+e^{-k-1}} T = I,$$

which may be written

$$(3.1) \quad S_{e^{-k-1}+e^{-1}} T S_e T S_{e^{-1}+e^{-k-1}} T S_{e^{-k+1}} T S_{e^{-k}} T = I.$$

We next apply (f') repeatedly as illustrated in the following sample computation.

$$S_{e^{-k-1}+e^{-1}} T S_e T S_{e^{-1}} T \cdot T^2 S_{e^{-k-1}} T S_{e^{-k+1}} T S_{e^{-k}} T = I,$$

$$T S_e T S_{e^{-1}} T S_{e^{k+2}-e^{k+1}+e} T S_{e^{-k-1}} T S_{e^{-k+1}} T S_{e^{-k}} T^2 = I,$$

$$(3.2) \quad S_{e^{k+2}-e^{k+1}+e} T S_{e^{-1}} T S_{e^{k+2}-e^{k+1}+e} T S_{e^{-k-1}} T S_{e^{-k+1}} T = I,$$

$$(3.3) \quad S_{e^{-k+1}+e^{-1}} T S_e T S_{e^{-1}+e^{-k-2}} T S_{e^{k+2}-e^{k+1}} T S_{e^{-k-1}} T = I,$$

$$(3.4) \quad S_{e^{-k-1}} T S_{e^{-1}} T S_{e^{k+2}-e^{k+3}+e} T S_{e^{-k-2}} T S_{e^{k+2}-e^{k+1}} T = I,$$

$$(3.5) \quad S_{e^{k+2}-e^{k+1}+e^{-1}} T S_e T S_{e^{-1}+e^{-k-3}} T S_{e^{k+2}-e^{k+3}} T S_{e^{-k-2}} T = I,$$

$$(3.6) \quad S_{e^{k+2}-e^{k+3}+e^{-1}} T S_{e^{-1}} T S_{e^{k+4}-e^{k+3}+e} T S_{e^{-k-3}} T S_{e^{k+2}-e^{k+1}} T = I,$$

$$(3.7) \quad S_{e^{k+2}-e^{k+3}+e^{-1}} T S_e T S_{e^{-k-4}} T S_{e^{k+4}-e^{k+3}} T S_{e^{-k-3}} T = I,$$

$$(3.8) \quad S_{e^{-k-3}} T S_{e^{-1}} T S_{e^{k+4}-e^{k+5}+e} T S_{e^{-k-4}} T S_{e^{k+4}-e^{k+3}} T = I.$$

These relations illustrate the four types that arise if the process is repeated indefinitely. It is evident that $S_{e^{p^n-1}} = S_1$ must appear. Suppose, for example, that $S_{e^{p^n-1}}$ appears in the following generalization of (3.4), say (3.2 u), where u is even. That is, we assume $u = p^n - k - 1$ in

$$(3.2 \cdot u) \quad S_{e^{-k-1}+e^{-1}} T S_{e^{-1}} T S_{e^{k+u}-e^{k+u+1}+e} T S_{e^{-k}} T S_{e^{k+u}-e^{k+u-1}} T = I$$

and easily reduce the left member to $S_0TS_{-1}(TS_1)^3S_{-1}T=I$ by means of (b) and (d).

3. **Proof of theorem.** Relations (a), (b), (d), and (f) are satisfied by

$$t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

which generate² $SLH(2, p^n)$. The corresponding form of R is

$$r = \begin{pmatrix} e^{-1} & 0 \\ 0 & e \end{pmatrix}$$

of period p^n-1 . The order l of L is not less than the order of $SLH(2, p^n)$. That is, $l \geq p^n(p^{2n}-1)$. Now

$$R^{-1}S_\lambda R = S_{-\lambda}TS_{-\lambda^{-1}}TS_{-\lambda}TS_\lambda TS_\lambda TS_{-\lambda^{-1}}T^3S_\lambda = S_{\lambda^2}$$

by (b) and (f'). Further, $R^{p^n-1}=I$ by (2) and (d). We conclude that $K=\{R, S_\lambda\}$ is of order $p^n(p^n-1)$ and all of its elements may be represented in either of the forms $R^\alpha S_\beta$, $S_\alpha R^\beta$. Now consider the p^n+1 sets of $p^n(p^n-1)$ elements represented by K , KTS_λ (λ arbitrary). There are at most $p^n(p^{2n}-1)$ distinct elements. It is evident that the sets are permuted among themselves on multiplication on the right by S_ρ . If $\lambda \neq 0$,

$$KTS_\lambda T = K(S_{-\lambda}TS_{-\lambda^{-1}}TS_{-\lambda}T)TS_\lambda T$$

by (2). Making obvious simplifications we obtain $KTS_\lambda T = KTS_{-\lambda^{-1}}$. Now $KTS_0T = KT^2 = K$, since $T^2 = (T^2S_{-1})^3$. Also $KT = KTS_0$. Hence the sets are also permuted among themselves on multiplication on the right by T . It follows that all the elements of L are in the sets and $l \leq p^n(p^{2n}-1)$. Hence L and $SLH(2, p^n)$ are of equal orders and simply isomorphic.

THEOREM 1. *The special linear homogeneous group $SLH(2, p^n)$, $p > 2$, of binary linear substitutions of determinant unity in the GF $[p^n]$ is simply isomorphic with the abstract group generated by the operators T and S_λ , where λ runs through the series of p^n marks of the field, subject to the generation relations*

- (a)³ $S_\lambda S_\mu = S_{\lambda+\mu}$ (λ, μ any marks),
- (b) $T^4 = I$, $S_\lambda T^2 = T^2 S_\lambda$,
- (d) $(S_1 T^3)^3 = I$,
- (e) $TS_\alpha TS_{2\alpha^{-1}} TS_\alpha TS_{2\alpha^{-1}} T^2 = I$ (α any mark $\neq 0$).

² Dickson, loc. cit., p. 80.

³ A referee has pointed out that $S_0 = I$ follows from $S_\lambda S_\mu = S_{\lambda+\mu}$.

THEOREM 2. *The special linear homogeneous group $SLH(2, 2^n)$ of binary linear substitutions of determinant unity in the $GF[2^n]$ is simply isomorphic with the abstract group generated by the operators T and S_λ , where λ runs through the series of 2^n marks of the field subject to the generational relations*

- (a) $S_\lambda S_\mu = S_{\lambda+\mu}$ (λ, μ any marks),
- (b) $T^4 = I, S_\lambda T^2 = T^2 S_\lambda$,
- (d) $(S_1 T^3)^3 = I$,
- (f) $TS_\alpha TS_{\alpha^{-1}} TS_\rho = S_{\alpha^{-2}\rho} TS_\alpha TS_{\alpha^{-1}} T$ (ρ arbitrary; $\alpha \neq 0$).

COROLLARY 1.⁴ *The linear fractional group $LF(2, p^n)$, $p > 2$, of linear fractional transformations in the $GF[p^n]$ is simply isomorphic with the abstract group generated by the operators T and S_λ , where λ runs through the series of p^n marks of the field, subject to the generational relations*

- (a) $S_\lambda S_\mu = S_{\lambda+\mu}$ (λ, μ any marks),
- (b') $T^2 = I$,
- (d') $(S_1 T)^3 = I$,
- (e') $(S_\alpha TS_{\alpha^{-1}} T)^2 = I$ (α any mark $\neq 0$).

COROLLARY 2. *The linear fractional group $LF(2, 2^n)$ of linear fractional transformations in the $GF[2^n]$ is simply isomorphic with the abstract group generated by the operators T and S_λ , where λ runs through the series of 2^n marks of the field, subject to the generational relations*

- (a) $S_\lambda S_\mu = S_{\lambda+\mu}$ (λ, μ any marks),
- (b') $T^2 = I$,
- (d') $(S_1 T)^3 = I$,
- (f) $TS_\alpha TS_{\alpha^{-1}} TS_\rho = S_{\alpha^{-2}\rho} TS_\alpha TS_{\alpha^{-1}} T$ (ρ arbitrary; $\alpha \neq 0$).

COROLLARY 3.⁵ *The abstract group $G_{p(p^2-1)/2}$ is simply isomorphic with the group $LF(2, p)$, $p > 2$, may be generated by two operators T and S subject to the generational relations*

$$S^p = T^2 = (ST)^3 = (S^r TS^{2/r})^3 = I, \quad r \neq 0.$$

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⁴ Special cases of this corollary have been proved by Dickson and Bussey. See the latter's dissertation, Proceedings of the London Mathematical Society, (2), vol. 3 (1905), pp. 296-315.

⁵ Due to W. H. Bussey, loc. cit., p. 303.

AN INTEGRAL ANALOGUE OF LAPLACE'S EQUATION¹

MAXWELL READE AND E. F. BECKENBACH

1. **Introduction.** A general problem of a formal nature may be stated, for functions of two variables, as follows. Let a class of functions $x(u, v)$, or a class of sets of functions $x_j(u, v)$, $j=1, 2, \dots, n$, defined in a finite simply connected domain D , be characterized as satisfying a set of one or more differential equations,

$$(1) \quad A = 0.$$

If in (1) we replace the partial derivatives

$$\frac{\partial y}{\partial u}, \quad \frac{\partial y}{\partial v}$$

by the line integrals

$$\frac{1}{\pi r^2} \int_{C_r} y dv, \quad - \frac{1}{\pi r^2} \int_{C_r} y du,$$

respectively, where C_r is an arbitrary circle, of variable radius r , lying in D , we obtain a set of integral equations,

$$(2) \quad B = 0,$$

analogous to (1). Our problem is the determination of the class of functions, or of sets of functions, characterized by (2).

If $y(u, v)$ has continuous first partial derivatives, then

$$\frac{1}{\pi r^2} \int_{C_r} y dv = y_u(u_0, v_0) + o(r^0), \quad - \frac{1}{\pi r^2} \int_{C_r} y du = y_v(u_0, v_0) + o(r^0),$$

where the subscripts u, v denote partial differentiation, where r and (u_0, v_0) are the length of the radius and the coordinates of the center of C_r , respectively, and where $o(r^\alpha)$ denotes a quantity (not always the same quantity) such that]

$$\lim_{r \rightarrow 0} \frac{o(r^\alpha)}{r^\alpha} = 0.$$

Hence, for functions for which the derivatives involved in (1) are continuous, (1) is equivalent to

$$B = o(r^0),$$

¹ Presented in part to the Society, June 20, 1940, by E. F. Beckenbach.

and a necessary condition that (2) hold for all circles in D is that (1) hold for all points in D ; we might consider (1) to be a special case of (2), corresponding to $r=0$.

Further local differential conditions, usually more restrictive than (1), but less severe than the integral condition (2), are obtained by replacing (2) by

$$(3) \quad B = o(r^\alpha)$$

for significant values of the exponent α . On the other hand, even more restrictive than (2) is

$$(4) \quad B^* = 0,$$

where B^* is obtained from B by replacing

$$\frac{1}{\pi r^2} \int_{C_r} \quad \text{by} \quad \int_\gamma,$$

where γ is an arbitrary closed rectifiable Jordan curve in D .

For example, if (1) is

$$x_{1,v} = x_{2,u},$$

the corresponding integral equation (4) is the familiar

$$\int_\gamma x_1 du + x_2 dv = 0.$$

Or if (1) is the set of Cauchy-Riemann differential equations,

$$x_{1,u} = x_{2,v}, \quad x_{1,v} = -x_{2,u},$$

the corresponding integral equations (4) are

$$\int_\gamma x_1 dv = - \int_\gamma x_2 du, \quad \int_\gamma x_1 du = \int_\gamma x_2 dv,$$

which can be written together as the Cauchy integral equation

$$(5) \quad \int_\gamma f(z) dz \equiv \int_\gamma [x_1(u, v) + ix_2(u, v)](du + idv) = 0.$$

Again, analogous to the differential conditions

$$\sum_{j=1}^3 x_{j,u}^2 = \sum_{j=1}^3 x_{j,v}^2, \quad \sum_{j=1}^3 x_{j,u} x_{j,v} = 0,$$

that is, $E=G$ and $F=0$, which characterize conformal, or isothermic,

maps, we have considered² the integral conditions

$$\sum_{j=1}^3 \left[\int x_j(u, v) du \right]^2 = \sum_{j=1}^3 \left[\int x_j(u, v) dv \right]^2,$$

$$\sum_{j=1}^3 \left[\int x_j(u, v) du \right] \left[\int x_j(u, v) dv \right] = 0.$$

The hierarchy of integral conditions discussed above leads this time to an integral characterization of isothermic maps; of isothermic spherical maps, both those which carry circles into circles and those which do not; of isothermic maps on minimal surfaces; and of isothermic plane maps. Moreover, the investigation has yielded³ unexpected information concerning mean-value surfaces.

2. Characterization of harmonic functions. In this paper we shall apply the above method to set up an integral equation analogous to Laplace's second order differential equation,

$$(6) \quad \Delta x \equiv \frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2} = 0,$$

which characterizes harmonic functions. This will illustrate how the method may be applied to equations of order higher than the first: we shall establish a second order integral equation characterizing harmonic functions. Our integral equation is not to be confused with other integral equations characterizing harmonic functions,

$$\int_{\gamma} \frac{dx}{dn} ds = 0,$$

$$x(u_0, v_0) = \frac{1}{2\pi} \int_0^{2\pi} x(u_0 + r \cos \phi, v_0 + r \sin \phi) d\phi,$$

$$x(u_0, v_0) = \frac{1}{\pi r^2} \iint_{r^2 + t^2 \leq r^2} x(u_0 + s, v_0 + t) ds dt;$$

rather, it seems to be more nearly analogous to the (first order) Cauchy integral equation (5) characterizing analytic functions of a complex variable. As in the case of the Cauchy integral equation,

² Maxwell Reade and E. F. Beckenbach, *Generalizations to space of the Cauchy and Morera theorems*, Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 92-97; also, Transactions of this Society, vol. 49 (1941), pp. 354-377.

³ Maxwell Reade and E. F. Beckenbach, *Mean-value surfaces*, to appear in an early issue of this Bulletin.

there is here no hierarchy of results: a weak condition of type (3) implies that $x(u, v)$ is harmonic, while the condition that $x(u, v)$ is harmonic implies a strong result of type (4).

We note that

$$(7) \quad \Delta x \equiv \bar{\lambda}(\lambda x) \equiv \lambda(\bar{\lambda}x),$$

where

$$\lambda \equiv \frac{\partial}{\partial u} + i \frac{\partial}{\partial v}, \quad \bar{\lambda} \equiv \frac{\partial}{\partial u} - i \frac{\partial}{\partial v}.$$

Let

$$C: \quad u = s(\tau), \quad v = t(\tau), \quad 0 \leq \tau \leq 1;$$

$$C^*: \quad u = \xi(\mu), \quad v = \eta(\mu), \quad 0 \leq \mu \leq 1,$$

be arbitrary closed rectifiable Jordan curves in the u, v -plane such that for each τ_0 , $0 \leq \tau_0 \leq 1$, the curve

$$C[s(\tau_0), t(\tau_0)]: \quad u = s(\tau_0) + \xi(\mu), \quad v = t(\tau_0) + \eta(\mu), \quad 0 \leq \mu \leq 1,$$

lies in D . Making the prescribed replacements in (6), in accordance with (7), we find the following integral analogues of Laplace's differential equation:

$$(8) \quad \int_C \left[\int_{C(s,t)} x(s + \xi, t + \eta)(d\xi - id\eta) \right] (ds + idt) = 0,$$

$$(9) \quad \int_C \left[\int_{C(s,t)} x(s + \xi, t + \eta)(d\xi + id\eta) \right] (ds - idt) = 0.$$

Now both (8) and (9) turn out to characterize harmonic functions; since the analysis is the same in the two cases, we shall confine our attention to just one of them, namely (8).

If (8) holds for fixed closed rectifiable Jordan curves C and C^* , then we shall say that (8) holds for the *curve-pair* (C, C^*) .

THEOREM 1. *If the real function $x(u, v)$ is continuous in a finite simply connected domain D , then a necessary and sufficient condition that $x(u, v)$ be harmonic in D is that*

$$(10) \quad \int_C \left[\int_{C(s,t)} x(s + \xi, t + \eta)(d\xi - id\eta) \right] (ds + idt) = 0$$

hold for all curve-pairs (C, C^*) for which the curves $C(s, t)$ all lie in D .

NECESSITY. If $x(u, v)$ is harmonic in D , and if $\xi(\mu)$ and $\eta(\mu)$ are the

coordinate functions of an arbitrary fixed closed rectifiable Jordan curve C^* , then

$$\int_{C(s,t)} x(s+\xi, t+\eta)(d\xi - i d\eta),$$

where $\{C(s, t)\}$ is the family of curves

$$C(s, t): u = s + \xi(\mu), \quad v = t + \eta(\mu), \quad 0 \leq \mu \leq 1,$$

defines a function $F(w)$ of the complex variable $w = s + it$; $F(w)$ is defined on a point-set $D(C^*)$ of the u, v -plane, consisting of finite open simply connected pieces. By Green's lemma, it follows that

$$(11) \quad F(w) = - \iint_{D(s,t)} [x_2(s+\xi, t+\eta) + ix_1(s+\xi, t+\eta)] d\xi d\eta,$$

where $D(s, t)$ denotes the interior of the curve $C(s, t)$, and where $x_1(u, v) \equiv \partial x / \partial u$, $x_2(u, v) \equiv \partial x / \partial v$. By Leibnitz' rule, we have

$$(12) \quad \begin{aligned} \frac{\partial}{\partial s} \iint_{D(s,t)} x_2(s+\xi, t+\eta) d\xi d\eta &= \iint_{D(s,t)} x_{2,2}(s+\xi, t+\eta) d\xi d\eta \\ &= \frac{\partial}{\partial t} \iint_{D(s,t)} x_1(s+\xi, t+\eta) d\xi d\eta, \end{aligned}$$

$$(13) \quad \frac{\partial}{\partial t} \iint_{D(s,t)} x_2(s+\xi, t+\eta) d\xi d\eta = \iint_{D(s,t)} x_{2,2}(s+\xi, t+\eta) d\xi d\eta,$$

$$(14) \quad \frac{\partial}{\partial s} \iint_{D(s,t)} x_1(s+\xi, t+\eta) d\xi d\eta = \iint_{D(s,t)} x_{1,1}(s+\xi, t+\eta) d\xi d\eta.$$

Since $x(u, v)$ is harmonic in D , (13) and (14) yield

$$(15) \quad \frac{\partial}{\partial t} \iint_{D(s,t)} x_2(s+\xi, t+\eta) d\xi d\eta = - \frac{\partial}{\partial s} \iint_{D(s,t)} x_1(s+\xi, t+\eta) d\xi d\eta.$$

From (11), (12) and (15) it follows that $F(w)$ is analytic in $D(C^*)$. Since for each closed rectifiable Jordan curve C lying in $D(C^*)$ the curve C encloses only points belonging to $D(C^*)$, it follows from Cauchy's theorem that (10) holds for (C, C^*) . Since C^* is an arbitrary closed rectifiable Jordan curve, it follows that (10) holds for all curve-pairs (C, C^*) for which the curves $C(s, t)$ all lie in D .

SUFFICIENCY. If $x(u, v)$ is continuous in D , then for a fixed positive ρ , the averaging function⁴

⁴ For a list of properties of averaging functions, see H. E. Bray, *Proof of a formula for an area*, this Bulletin, vol. 29 (1923), pp. 264-270.

$$x(u, v; \rho) \equiv \frac{1}{\pi \rho^2} \iint_{\xi^2 + \eta^2 \leq \rho^2} x(u + \xi, v + \eta) d\xi d\eta$$

has continuous partial derivatives of the first order; of course, $x(u, v; \rho)$ can be defined thus only for an open subset D_ρ of D , but this is of no consequence since ρ is arbitrarily small. We have the following equation

$$(16) \quad \bar{\lambda}x(u, v; \rho) = \frac{i}{\pi \rho^2} \iint_{\xi^2 + \eta^2 \leq \rho^2} x(u + \xi, v + \eta)(d\xi - i d\eta).$$

If (10) holds, then it follows from (16) and the theorem of Morera that $\bar{\lambda}x(u, v; \rho)$ is analytic in D_ρ , and hence it follows from the Cauchy-Riemann equations that $x(u, v; \rho)$ is harmonic in D_ρ . Since in any closed subset of D , $x(u, v; \rho) \rightarrow x(u, v)$ as $\rho \rightarrow 0$, it follows that $x(u, v)$ is harmonic in D .

3. Lessening the sufficiency conditions. Actually, we need a good deal less to prove $x(u, v)$ harmonic in D . In the first place, letting $\Im \phi$ denote the imaginary part of the complex function ϕ , we have the following result.

THEOREM 2. *If the real function $x(u, v)$ is continuous in a finite simply connected domain D , then*

$$(17) \quad \Im \int_C \left[\int_{C(s, t)} x(s + \xi, t + \eta)(d\xi - i d\eta) \right] (ds + i dt) = 0$$

holds for all curve-pairs (C, C^*) for which the curves $C(s, t)$ all lie in D .

PROOF. The repeated average,

$$x(u, v; \rho^{(2)}) \equiv \frac{1}{\pi \rho^2} \iint_{\beta^2 + \zeta^2 \leq \rho^2} x(u + \beta, v + \zeta; \rho) d\beta d\zeta,$$

has continuous partial derivatives of the second order. We have

$$\begin{aligned} \Im \int_C \left[\int_{C(s, t)} x(s + \xi, t + \eta; \rho^{(2)})(d\xi - i d\eta) \right] (ds + i dt) \\ = \int_C \left[\int_{C(s, t)} x(s + \xi, t + \eta; \rho^{(2)}) d\xi \right] dt \\ - \int_C \left[\int_{C(s, t)} x(s + \xi, t + \eta; \rho^{(2)}) d\eta \right] ds; \end{aligned}$$

hence, by Green's lemma and Leibnitz' rule, we obtain

$$(18) \quad \Im \int_C \left[\int_{C(s,t)} x(s + \xi, t + \eta; \rho^{(2)}) (d\xi - id\eta) \right] (ds + idt) \\ = \iint_{D(C)} \left\{ \iint_{D(s,t)} \left[- \frac{\partial^2 x(s + \xi, t + \eta; \rho^{(2)})}{\partial s \partial \eta} \right. \right. \\ \left. \left. + \frac{\partial^2 x(s + \xi, t + \eta; \rho^{(2)})}{\partial t \partial \xi} \right] d\xi d\eta \right\} ds dt,$$

where $D(C)$ denotes the interior of C and where $D(s, t)$ denotes the interior of $C(s, t)$. From (18) it follows that

$$(19) \quad \Im \int_C \left[\int_{C(s,t)} x(s + \xi, t + \eta; \rho^{(2)}) (d\xi - id\eta) \right] (ds + idt) = 0$$

holds for all curve-pairs (C, C^*) for which the curves $C(s, t)$ all lie in the region of definition of $x(u, v; \rho^{(2)})$. Since in any closed subset of D , $x(u, v; \rho^{(2)}) \rightarrow x(u, v)$ as $\rho \rightarrow 0$, we obtain (17) from (19).

It follows that *harmonic functions are characterized by the vanishing of only the real part of the left-hand member of (10)*.

In the second place, we can replace the strong integral condition (10), which is of type (4), by a weaker point condition of type (3). Let k be a fixed positive constant, let $C(u_0, v_0; r)$ denote an arbitrary circle in D with center at (u_0, v_0) and of radius r , and let $C(u_0 + s, v_0 + t; kr)$ denote the circles in D having common radii kr and having their centers on the circumference of the circle $C(u_0, v_0; r)$. If $\Re\phi$ denotes the real part of the complex function ϕ , then we have the following result.

THEOREM 3. *If the real function $x(u, v)$ is continuous in a finite simply connected domain D , then a necessary and sufficient condition that $x(u, v)$ be harmonic in D is that for each point (u_0, v_0) in D ,*

$$(20) \quad \Re \int_{C(u_0, v_0; r)} \left[\int_{C(u_0 + s, v_0 + t; kr)} x(u_0 + s + \xi, v_0 + t + \eta) \right. \\ \left. \cdot (d\xi - id\eta) \right] (ds + idt) = o(r^4),$$

the integrals being taken once around the circles in the positive (counter-clockwise) sense.

PROOF. If it is given that (20) holds, then also

$$(21) \quad \Re \int_{C(u_0, v_0; r)} \left[\int_{C(u_0+s, v_0+t; kr)} x(u_0+s+\xi, v_0+t+\eta; \rho^{(2)}) \cdot (d\xi - id\eta) \right] (ds + idt) = o(r^4).$$

A computation yields

$$(22) \quad \Re \int_{C(u_0, v_0; r)} \left[\int_{C(u_0+s, v_0+t; kr)} x(u_0+s+\xi, v_0+t+\eta; \rho^{(2)}) \cdot (d\xi - id\eta) \right] (ds + idt) = \pi^2 k^2 r^4 \Delta x(u_0, v_0; \rho^{(2)}) + o(r^4).$$

By (21) and (22), $x(u, v; \rho^{(2)})$ is harmonic in its region of definition, whence, as above, $x(u, v)$ is harmonic in D .

If, conversely, it is given that $x(u, v)$ is harmonic, then we obtain (20) from an expansion similar to (22) for the function $x(u, v)$.

We note that an alternate proof of Theorem 1 might be given along the line of the above proof of Theorem 2, and that the proof here given of Theorem 3 contains still another proof of the sufficiency part of Theorem 1.

OHIO STATE UNIVERSITY AND
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THE CONSTRUCTION OF POSITIVE TERNARY QUADRATIC FORMS¹

GORDON PALL

1. **Binary quadratic forms.** A positive, binary quadratic form $f = (a, t, b) = ax^2 + 2txy + by^2$ (a, t , and b real numbers) is equivalent to one and only one form in which

$$(1) \quad |2t| \leq a \leq b, \quad t \geq 0 \text{ if } a = b \text{ or } a = |2t|.$$

Such a form is called *reduced*, and has the further properties that: (i) a is the least number properly represented by f ; (ii) $ab \leq 4\Delta/3$, where $\Delta = ab - t^2$; (iii) among all forms equivalent to f with the minimum a as first coefficient, b is the least possible.

To obtain all reduced, classical, binary quadratic forms of a given determinant Δ , we have the following well known algorithm: factor $\Delta + t^2$ ($\pm t = 0, 1, \dots, (\Delta/3)^{1/2}$) as ab , with $|2t| \leq a \leq b$, in all possible ways; but discard forms with $t < 0$ if $a = b$ or $|2t|$.

2. **Ternaries.** We develop in this article a similar method of finding a unique form in every class of integral ternary quadratic forms of a *given* determinant, or in a given order or genus. The methods of reduction hitherto devised (Seeber [1], Eisenstein [2], Selling [3]) are entirely adequate if one wishes to calculate all reduced forms of determinant less than a certain fixed value, but are not connected closely enough with the invariants to make the computation of forms with given values for their invariants practicable. By the method of this article one can obtain the reduced forms in a given genus of determinant around 1000 in fifteen minutes.

Our reduced form is not the same as that of Eisenstein or Selling, but we shall see how to pass from our form to that of Eisenstein.

Eisenstein found that within every class of real, positive, ternary quadratic forms $f = (a, b, c, r, s, t) = ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy$ there is a unique form (to be called *E-reduced*) satisfying the following inequalities:

- (2) r, s, t all > 0 , or all ≤ 0 ;
- (3) $a \geq 2|s|, a \geq 2|t|, b \geq 2|r|$;
- (4) $a \leq b \leq c; a + b + 2r + 2s + 2t \geq 0$;
- (5) if $a = b, |r| \leq |s|$; if $b = c, |s| \leq |t|$;

¹ Presented to the Society, May 2, 1941.

- (6) if $a + b + 2r + 2s + 2t = 0$, $a + 2s + t \leq 0$;
 (7) if $a = 2t$, $s \leq 2r$; if $a = 2s$, $t \leq 2r$; if $b = 2r$, $t \leq 2s$;
 (8) if $a = -2t$, $s = 0$; if $a = -2s$, $t = 0$; if $b = -2r$, $t = 0$.

Here the conditions (2), (3), and (4) are to be regarded as the principal inequalities, and determine the planar boundaries of the fundamental cell of reduced forms. From the principal inequalities readily follows Seeber's inequality $abc \leq 2d$, where d is the determinant of f .

The number of possibilities for a given d is greatly limited; but no procedure is indicated for sifting out forms of a given determinant.

It may be observed that in an E -reduced form a is the minimum, and, among all forms equivalent to f , b is as small as it can be for the predetermined a , and c as small as it can be for the preceding a and b .

Eisenstein used two methods in constructing his table of integral ternaries. One was to write down sequences of forms satisfying (2)–(8), and to calculate their determinants. The other produced all forms of a given determinant with minimum $a = 1, 2$, or 3 ; but (2)–(8) are ill adapted to extend this method to $a > 3$. This article provides such an extension.

Let $F = (A, B, C, R, S, T)$ be the adjoint of f . On account of the rarity of Seeber's book it is worth noting that Seeber's conditions differ from Eisenstein's only in having (6), (7), and (8) replaced by

- if $a + b + 2r + 2s + 2t = 0$, $2 \mid R \mid \leq C$; if $a = \pm 2t$, $2 \mid T \mid \leq B$;
 if $a = \pm 2s$, $2 \mid S \mid \leq C$; if $b = \pm 2r$, $2 \mid R \mid \leq C$.

The equivalence of these conditions to (6)–(8) is obvious from identities like those we use in §4. It may be worth noting in regard to our method that while certain coefficients of F start off our process of construction (§7), only linear expressions in the coefficients of f need to be formed in testing for duplicates.

3. A new reduced form. We call the real, positive ternary quadratic form $f = (a, b, c, r, s, t)$ *reduced*, if the following inequalities hold:

- (2) r, s, t all > 0 , or all ≤ 0 ;
 (9) $a \leq b$, $a \leq c$, $a \leq b + c - 2r$, $0 \leq b + c + 2r + 2s + 2t$;
 (10) $a \geq 2 \mid s \mid$, $a \geq 2 \mid t \mid$;
 (11) $2 \mid R \mid \leq C \leq B$, where $R = st - ar$, $C = ab - t^2$, $B = ac - s^2$;
 (12) if $a = -2s$ or $-2t$, then t or $s = 0$;
 (13) if $a = 2s$ or $2t$, then $R \leq 0$ (i.e., t or $s \leq 2r$, resp.);

- (14) if $C = B$, $b \leq c$ (hence $s^2 \geq t^2$); if $C = -2R$, $b \geq 2r$;
 if $C = 2R$, $a + 2s + t \geq 0$;
 (15) if $a = b$, $|r| \leq |s|$; if $a = c$, $|r| \leq |t|$;
 (16) if $a = b$ or c , then $a + r + s + t \geq 0$;
 (17) if $a = b + c - 2r$ then $b \leq r + t$;
 (18) if $b + c + 2r + 2s + 2t = 0$, then $b + 2t + r \leq 0$ and $b \leq c$.

It should be observed that (9₄), (12), (14₃), (16), and (18) are trivial if (2₁) holds; and (9₃), (13), (14₂), and (17) are trivial if (2₂) holds.

THEOREM 1. *Every class of real, positive, ternary quadratic forms contains one and only one reduced form.*

We prove in this section that every class contains a reduced form. Denote the adjoint of f by $F = (A, B, C, R, S, T)$, so that $C = ab - t^2$ and $BC - R^2 = ad$. We can replace f by an equivalent form in which:

- (19) a is the minimum of f ; C is the least possible with the aforesaid a ;
 and B is the least possible with the predetermined a and C .

We have $2|R| \leq C \leq B$ by (19₂) and (19₃). For replacing y by $y \mp z$ in f replaces a by a , C by C , B by $C \pm 2R + B$.

Replacing x by $x + hy + kz$ in f corresponds in the adjoint transformation to replacing y by $y - hx$ and z by $z - kx$ in F , hence leaves C , R , and B unchanged but replaces s by $s + ak$, t by $t + ah$. We thus secure (10) and (11).

We get (2) by changing signs of an even number of x, y, z .

We next show that conditions (9) and (15)–(18) all follow from (19). First notice that (19₁) implies (9): $a \leq f(0, 1, 0), \dots, f(1, 1, 1)$.

Denote by (X, Y, Z) the unimodular transformation replacing x, y, z by X, Y, Z . If $a = b$, use $(-y, -x, -z)$ and get (a, a, c, s, r, t) ; if $a = c$, $(-z, -y, -x)$ and (a, b, a, t, s, r) . Hence (19₃) and (19₂) imply that $ac - r^2 \geq ac - s^2$, and $ab - r^2 \geq ab - t^2$, or (15).

If $a = b - 2r + c$, the transformation $(z, -x, x - y)$ carries f into $(b - 2r + c, c, a, -s, s - t, r - c)$. Hence $(b - 2r + c)c - (r - c)^2 = bc - r^2 = b(a + 2r - b) - r^2 = ab - (b - r)^2 \geq ab - t^2$, giving (17).

If $a = b$, $(y - z, x - z, -z)$ replaces f by $(a, a, \sigma, -a - t - s, -a - t - r, t)$, where $\sigma = a + b + c + 2r + 2s + 2t$; if $a = c$, $(-y + z, -y, x - y)$ replaces f by $(a, \sigma, a, -a - s - t, s, -a - r - s)$. Hence $a\sigma - (a + t + r)^2 \geq ac - s^2$, or $(a + s)^2 \geq (r + t)^2$; and $a\sigma - (a + r + s)^2 \geq ab - t^2$, or $(a + t)^2 \geq (r + s)^2$; giving (16).

Let $b + c + 2r + 2s + 2t = 0$. Applying $(-x + z, -x + y, -x)$ and

$(x, x-z, x+y)$, we get $(a, b, a, t, -a-t-s, -t-b-r)$ and $(a, c, b, -r, -b-r-t, c+r+s)$. By (19₂), $ab-(t+b+r)^2 \geq ab-t^2$, or $b+2t+r \leq 0$. Also, $ac-(c+r+s)^2 \geq ab-t^2$, or $t^2-(c+r+s)^2 \geq a(b-c) = a(-2c-2r-2s-2t)$; if b could exceed c , then $-c-r-s-t = b+r+s+t > 0$, whence $c+r+s-t \geq 2a$, while $c < -r-s-t$; thus $-r-s-t > 2a-r-s+t$, $-2t > 2a$, contrary to (10). This gives (18).

If $a = -2s$ and $t \neq 0$, $(x-z, y, -z)$ carries f into $f_1 = (a, b, c, -t-r, -s, -t)$, with $R_1 = -R$, $C_1 = C$, $B_1 = B$. Here $(-s)(-t) < -2s(-t-r)$. Hence (13) holds for f_1 . If $a = -2t$ and $s \neq 0$, use $(-x-y, -y, z)$ and $(a, b, c, -r-s, -s, -t)$.

If $a = 2s$ and $R = s(t-2r) > 0$, $(-x-z, -y, z)$ yields $(a, b, c, t-r, s, t)$, with $t-2(t-r) < 0$. If $a = 2t$ and $R > 0$, use $(-x-y, y, -z)$ and $(a, b, c, s-r, s, t)$.

If $C = B$, we may replace f by (a, c, b, r, t, s) , and thus secure (14₁). This does not affect any conditions so far obtained.

If $C = -2R$ and $b < 2r$, $(-x, -y-z, z)$ replaces f by $f_2 = (a, b, b-2r+c, b-r, t-s, t) \sim (a, b, b-2r+c, r-b, t-s, -t)$, with $R_2 = -R-C = R$, $B_2 = B$, and $b \geq 2(b-r)$. The minimal conditions (9) and (15)-(18) must hold in consequence of (19); and (10), (11) also hold, for f_2 . If $a = 2t$ and $s = t$ we have (12); if $a = 2t$ and $s < t$, $R_2 = R \leq 0$ by (13) for f . If also $C = B$, and $c < 2r$, we use $(a, b-2r+c, b, b-r, t, t-s)$, and have $b-2r+c \geq 2(b-r)$ since $b \leq c$ by (14₁) for f .

Finally let $C = 2R$, $a+2s+t < 0$. Then $(-x+z, -y+z, z)$ carries f into $(a, b, \sigma, -t-b-r, -a-t-s, t)$, with C, R , and B unchanged. Here $a \geq 2(a+t+s)$ since $a+2s+2t < 0$; and we cannot have $a = -2t$ since then $s = 0$ by (12) for f . But $a+2(-a-t-s)+t = -(a+2s+t) > 0$. If also $C = B$ and $a+c+2r+2s+2t < 0$, use $(a, \sigma, b-t-b-r, t, -a-t-s)$, where $a+2t+(-a-t-s) = t-s \geq 0$ by (14₁) for f .

4. Some properties of a reduced form. Before completing the proof of Theorem 1, we note further inequalities for a reduced form:

$$(20) \quad \text{if } b = 2r, t \leq 2s; \text{ if } b < 2r, t < 2s$$

$$(21) \quad \text{if } c = 2r, s \leq 2t; \text{ if } c < 2r, s < 2t$$

$$[\text{for, } 0 \leq C + 2R = a(b-2r) + t(2s-t), B + 2R \text{ similarly}];$$

$$\text{if } r, s, t \leq 0, \text{ then } b \geq -2r \text{ and } c \geq -2r;$$

$$(22) \quad \text{if } b = -2r, t = 0; \text{ if } c = -2r, s = 0$$

$$[\text{for, } 0 \leq C - 2R = a(b+2r) - t(t+2s), B - 2R \text{ similarly}];$$

$$(23) \quad \text{if } r, s, t > 0, \text{ then } 5b > 8r \text{ and } 5c > 8r$$

[for, $C > -2R$ by (14₃), $s^2 - (t-s)^2 > a(2r-b)$, $ab \geq 4s^2$, $b > 4(2r-b)$; $B \geq -2R$, $c \geq 4(2r-c)$, equality implying $s=t$, $c \geq b$];

$$(24) \quad c + |t| \geq b + |s|;$$

if $b = c$, $|s| \leq |t|$; if $b > c$, $|s| < |t|$

[for, $0 \leq B - C = (t-s)(t+s) - a(b-c) \leq a(c-b \pm (t-s))$];

$$(25) \quad \text{if } r, s, t > 0, c \geq r + s$$

[for, $a \leq b - 2r + c$, $c - b \geq s - t$, $c - (a + 2r - c) \geq s - t$, $2c \geq 2r + a + s - t \geq 2r + 2s$];

$$(26) \quad \text{if } r, s, t \leq 0, c + 2s + r \geq \frac{1}{2}(b - c)$$

[for, by (9₄), $c + 2s + r \geq c + 2s - \frac{1}{2}b - \frac{1}{2}c - s - t = s - t - \frac{1}{2}(b - c)$];

$$(27) \quad \text{if } a + b + 2r + 2s + 2t = 0, C = 2R \text{ if and only if } a + 2s + t = 0,$$

and $C > 2R$ if and only if $a + 2s + t < 0$;

if $a + b + 2r + 2s + 2t < 0$, then $a + 2s + t < 0$

[for, $C - 2R = a(a + b + 2r + 2s + 2t) - (a + t)(a + 2s + t)$];

$$(28) \quad \text{if } a + c + 2r + 2s + 2t = 0, B \geq 2R \text{ if and only if } a + 2t + s \leq 0$$

[for, $B - 2R = a(a + c + 2r + 2s + 2t) - (a + s)(a + 2t + s)$].

We note that we cannot have $C = 2R$ if $r, s, t > 0$, nor $C = -2R$ if $r, s, t \leq 0$. Thus if $r, s, t > 0$, $ab > \frac{1}{2}ab + \frac{1}{2}ab \geq t^2 + 2st - 2ar$.

5. The equivalent Eisenstein-reduced form. We shall now exhibit for any reduced $f = (a, b, c, r, s, t)$ the equivalent E -reduced form; and Theorem 1 will follow when we verify that no such E -reduced form is equivalent to two distinct reduced forms. Each of the following fourteen forms is obtained from f by an unimodular transformation, respectively: (x, y, z) , $(-x, -z, -y)$, $(-x, -y-z, z)$, $(x, y+z, -y)$, $(x, -y+z, -z)$, $(-x, -y+z, +y)$, $(-x, z, y-z)$, $(x, -y, y-z)$, (x, y, z) , $(-x, -z, -y)$, $(-x+z, -y+z, z)$, $(x-z, -z, y-z)$, $(x-y, -y+z, -y)$, $(-x+y, y, y-z)$.

	Case $r, s, t > 0$	E -reduced form equivalent to f
1°	$2r \leq b \leq c$	(a, b, c, r, s, t)
2°	$2r \leq c < b$	(a, c, b, r, t, s)

	Case $r, s, t > 0$	E -reduced form equivalent to f
3°	$b < 2r \leq c, s < t$	$(a, b, b-2r+c, b-r, t-s, t)$
4°	$b \leq c < 2r, s < t$	$(a, b-2r+c, b, b-r, t, t-s)$
5°	$b < 2r \leq c, t \leq s$	$(a, b, b-2r+c, r-b, t-s, -t)$
6°	$b \leq c < 2r, t \leq s$	$(a, b-2r+c, b, r-b, -t, t-s)$
7°	$c < 2r \leq b$	$(a, c, b-2r+c, r-c, s-t, -s)$
8°	$c < b < 2r$	$(a, b-2r+c, c, r-c, -s, s-t)$

Use the abbreviation $\sigma = a+b+c+2r+2s+2t$.

	Case $r, s, t \leq 0$	E -reduced form equivalent to f
9°	$b \leq c, \sigma - c \geq 0$	(a, b, c, r, s, t)
10°	$c < b, \sigma - b \geq 0$	(a, c, b, r, t, s)
11°	$\sigma - c < 0, \sigma - b \geq 0$	$(a, b, \sigma, -t-b-r, -a-t-s, t)$
12°	$\sigma - c \geq 0, \sigma - b < 0$	$(a, c, \sigma, -s-c-r, -a-t-s, s)$
13°	$b \leq c, \sigma - c < 0, \sigma - b < 0$	$(a, \sigma, b, -t-b-r, t, -a-t-s)$
14°	$c < b, \sigma - c < 0, \sigma - b < 0$	$(a, \sigma, c, -c-r-s, s, -a-t-s)$

We first verify that the forms in the last column satisfy the conditions of E -reduction. We cannot have $a=b$ in 3°–6°, nor $a=c$ in 7°–8°, since then by (15) $r \leq s \leq \frac{1}{2}b$, or $r \leq t \leq \frac{1}{2}c$. Nor can $a=b-2r+c$ in 7°–8°, for then $c < b \leq r+t$, $a < r+t-2r+r+t=2t$. We use (23) for (2) in 3°–8°; (10) for (3); (9) for (4₁); (10) for (4₂) in 5°–8°, (6) being vacuous in 5°, 7°, and 8°; (14) and (24) in 1°–2°, (16) in 3°–6°; as to (6) and (8) in 6°, if $b=c, s=t$ by (24); we use (13) and (20)–(21) for (7) in 1°–2°, (23) ($t-s < r < 2b-2r$) in 3°–4°; if $a=2t$ in 5°–6°, or $a=2s$ in 7°–8°, then $a \geq 2s \geq 2t$ or $a \geq 2t \geq 2s, s=t$.

All of 9°–14° satisfy (2₁); for by (10) and (22), $b \geq -r-t, c \geq -r-s$, and $a \geq -s-t$. All satisfy (3) by (11) and (22). For (4₁) we have (9). As to (4₂) the sum $a+b+\dots$ is respectively $\sigma-c, \sigma-b, c-\sigma, b-\sigma, c-b$, and $b-c$; thus (6) follows for 9°–10° from (27)–(28), for 13° from (24₂), and is vacuous for the rest. As to (5), use: (14) and (24) in 9°–10°; (18) in 13°–14°; (27) or (28) in 11°–12°, noting that $a=b$ (or $a=c$) is excluded by (16). As to (8): use (12) and (22) in 9°–10°; if $a=-2t$ in 11°–12°, $s=0$ by (12), $b < -2r$, contrary to (22); if $a+2s+2t=0$ in 11°–14°, b (or c) $< -2r$, a contradiction; if $b+2t+2r=0$ in 11°, $a < -2s$; if $c+2r+2s=0$ in 12°, $a < -2t$; $a \neq -2s$ in 12° and 14°, since $-s < -t$ by (24); in 13°, $\sigma=2b+2t+2r$ implies $a+2s+c-b=0, s=t$ by (24), $a=-s-t$; and finally, in 14°, $\sigma=2c+2r+2s$ implies $a+2t+b-c=0$, a contradiction.

It is easy to verify that in terms of the quantities C, R, B of f , the values of C, R, B of the forms 1° – 14° are as follows:

$$\begin{aligned}
 & (C, R, B), \quad (B, R, C), \quad (C, -C - R, C + 2R + B), \\
 & \quad (C + 2R + B, -C - R, C), \\
 & (C, R + C, C + 2R + B), \quad (C + 2R + B, R + C, C), \\
 (29) \quad & (B, R + B, C + 2R + B), \quad (C + 2R + B, R + B, B), \\
 & (C, R, B), \quad (B, R, C), \quad (C, C - R, C - 2R + B), \\
 & \quad (B, B - R, C - 2R + B), \\
 & (C - 2R + B, C - R, C), \quad (C - 2R + B, B - R, B).
 \end{aligned}$$

6. **No two reduced forms are equivalent.** Denote two reduced forms by f and $f' = (a', b', c', r', s', t')$, the adjoint of f' by (A', B', C', R', S', T') , and let i° denote i° for f' . A moment's examination shows that for any i ($1 \leq i \leq 14$) $i^\circ = i'^\circ$ implies in some order that $a = a'$, $b = b'$, $\dots$, $t = t'$. We shall obtain a contradiction if $i^\circ = j'^\circ$ with $i \neq j$.

Clearly forms 1° to 4° , with all terms positive, cannot coincide with any of forms $5'^\circ$ to $14'^\circ$.

If $5^\circ = 11'^\circ$ (or $6^\circ = 13'^\circ$), $a = a'$, $-t = t'$, $t - s = -a' - t' - s'$; thus $-2s' \leq a' = s - s' = a \geq 2s$, $s = -s'$, $a' = -2s'$, $t' = 0$, but $t > 0$. Similarly by interchanging b and c , s and t , for $7^\circ = 12'^\circ$ and $8^\circ = 14'^\circ$.

In all other cases we shall obtain one of the contradictory conclusions (i) $B = C$ but $b > c$ or $s^2 < t^2$; (ii) $C = -2R$ but $b < 2r$; or (iii) $C = 2R$ but $a + 2s + t < 0$; for either f or f' .

If $i^\circ = j'^\circ$ the i th and j th triples in (29) for f and f' respectively must coincide. Thus if $3^\circ = 4'^\circ$, $C = C' + 2R' + B' \geq B' \geq C' = C + 2R + B \geq B \geq C$. In a similar way we find $B = C$ and $B' = C'$ for the pairs $(i, j) = (1, 2), (1, 4), (2, 3), (3, 4), (5, 7), (5, 8), (5, 10), (5, 12), (6, 8), (6, 14), (7, 11), (7, 13), (8, 13), (9, 10), (10, 11), (11, 12), (13, 14)$; all these cases give (i). The same method yields (ii) in the cases $(5, 13), (5, 14), (6, 7), (6, 11), (6, 12), (7, 8), (7, 10), (8, 9), (8, 11), (8, 12)$.

If $1^\circ = 3'^\circ$, (29) gives $C = C'$, $-R = C' + R' \geq -R' = C' + R \geq -R$, hence $C' = -2R'$, but $b' < 2r'$; similarly for $(2, 4), (5, 6), (5, 9), (6, 9), (6, 10), (8, 10)$. The same method leads to (i) in $(7, 9), (9, 12)$ [$R = B' - R' \geq R' = B' - R = C - R \geq R$, $C = 2R$, $B' = 2R'$, $C' = B'$, but $b' > c'$], $(9, 14), (10, 12), (10, 14)$, and $(12, 13)$.

Finally, $(7, 14), (9, 11), (9, 13), (10, 13), (11, 13), (11, 14)$, and $(12, 14)$ lead to (iii). Thus in $(7, 14)$, $B = C' - 2R' + B'$, $R + B = B' - R'$, $C + 2R + B = B'$, $B \geq B' \geq B$, $C' = 2R'$, $a' + 2s' + t' \geq 0$ by (14), whereas in $14'^\circ$, $s' - c' = a' + b' + 2r' + 2s' + 2t' < 0$, or by (27), $a' + 2s' + t' < 0$.

Several cases could have been contradicted in more than one way.

7. **Algorithm for forms of determinant d .** The minimum a of f , by Seeber's inequality, satisfies $a^2 \leq 2d$. Further, a is the minimum of the binary form (a, t, b) of determinant C , so that $a^2 \leq 4C/3$; and C is the minimum of (C, R, B) of determinant ad , whence $C^2 \leq 4ad/3$.

We are now ready to state an extension to ternaries of the binary algorithm of §1. We shall confine the statement to classical forms although it is easy to include forms in which $2r$, $2s$, or $2t$ may be odd.

Algorithm. Give a the values $1, 2, \dots, [(2d)^{1/2}]$, $\pm R$ the values $0, 1, \dots, [(\frac{1}{3}ad)^{1/2}]$. For each a and R , factor $ad + R^2$ in all ways as a product CB of positive integers C, B for which the congruences

$$(30) \quad C \equiv -t^2, \quad B \equiv -s^2 \pmod{a}$$

are solvable, $2|R| \leq C \leq B$, and $\frac{3}{4}a^2 \leq C$. Give s and t all solutions of (30) such that $|s| \leq \frac{1}{2}a$ and $|t| \leq \frac{1}{2}a$, and construct the quotients $b = (C + t^2)/a$ and $c = (B + s^2)/a$, and $r = (st - R)/a$, discarding cases in which r is not an integer [r will usually be an integer by choice of the sign of R , since $R^2 \equiv (st)^2 \pmod{a}$]. Change signs of two of r, s, t if necessary to secure (2). Discard all the resulting forms (a, b, c, r, s, t) which do not satisfy (9) and (12)–(18), as they are duplicates.

The forms remaining comprise one and only one representative of every (classical) class of determinant d . The Eisenstein reduced forms, if desired, may be written down by means of 1° to 14° in §5.

8. **Contracted process for an order or genus.** If we desire only the properly or improperly primitive (p.p. or i.p.) forms of an order (Ω, Δ) , then in §7, the values of C, B, R are multiples of Ω , and we can set

$$(31) \quad C = \theta\Omega C_1, \quad B = \theta\Omega B_1, \quad R = \Omega R_1,$$

where $\theta = 1$ or 2 according as the reciprocal forms are to be p.p. or i.p. If f is to be i.p., a, b , and c must be even.

There is not much lessening of labor over that for the whole determinant if $\Omega = 1$ (and, if $\Delta = 1$, it is best to use the reciprocal order). But if Ω and Δ exceed 1, the possible values of C within the limits $\frac{3}{4}a^2$ and $(4ad/3)^{1/2}$ (where $d = \Omega^2\Delta$), are comparatively few; and it is less work to write out the values of C for each a , using (31₁) and (30₁), and to construct the equations $ad + R^2 = CB$ by solving the congruences $a\Delta + R_1^2 \equiv 0 \pmod{\theta^2 C_1}$ and forming $a\Delta + R_1^2 = \theta^2 C_1 B_1$, with $2|R_1| \leq \theta C_1 \leq \theta B_1$ and with (30₂) solvable for $B = \theta\Omega B_1$. When a is not prime to 2Ω it is necessary to calculate the remaining cofactors A, S , and T , to test whether the form belongs to the desired order.

A specification of the numbers representable properly by the order, and the reciprocal order (and this is especially true within the nar-

rower confines of a genus) will diminish considerably the number of pairs of values of a and C to be considered. We shall not take the space here to list these numbers; however, compare Example 2.

9. **Example 1.** There are over 175 classical, reduced forms of determinant $d=600$. We shall find all with minimum $a=4$. Using a factor table we form $2400+R^2=CB$, for $\pm R=0, 1, \dots, 28$, with $12 \leq C \leq (3200)^{1/2}$ and $2|R| \leq C \leq B$, C and B congruent to 0 or 3 (mod 4). [For somewhat larger values of a it would be less work to write out the admissible values of C and to solve the congruences $-ad \equiv R^2 \pmod{C}$ for R .] We may take $t=0$ or ± 2 if $C \equiv 0$, $t = \pm 1$ if $C \equiv 3 \pmod{4}$; and similarly for s and B . We thus obtain the following table:

$\pm R$	C	B	a	b	c	r	s	t	a	b	c	r	s	t	
0	12	200	4	4	50	0	0	-2	4	4	51	1	2	2	(α)
0	15	160	4	4	40	0	0	-1							(β)
0	20	120	4	5	30	0	0	0	4	5	31	0	-2	0	
			4	6	30	0	0	-2	4	6	31	1	2	2	
0	24	100	4	6	25	0	0	0	4	6	26	0	-2	0	
			4	7	25	0	0	-2	4	7	26	1	2	2	
0	32	75	4	8	19	0	-1	0							(γ)
0	40	60	4	10	15	0	0	0	4	10	16	0	-2	0	
			4	11	15	0	0	-2	4	11	16	1	2	2	
4	16	151	4	4	38	-1	-1	0							(γ)
6	12	103	[4	4	51	2	1	2]							(α), (γ), (δ)
6	28	87	4	8	22	2	1	2							(γ), (ϵ)
7	31	79	4	8	20	2	1	1							
8	28	88	4	7	22	-2	0	0	4	7	23	-2	-2	0	
			4	8	22	-2	0	-2	4	8	23	3	2	2	(γ)
8	44	56	4	11	14	-2	0	0	4	11	15	-2	-2	0	
			4	12	14	-2	0	-2	4	12	15	3	2	2	(γ)
14	44	59	4	12	15	4	1	2							(γ), (ϵ)
15	35	75	4	9	19	4	1	1							
16	32	83	4	8	21	-4	-1	0							(ζ)
25	55	55	4	14	14	-6	-1	-1							

We have here omitted: (α) $t=0$ since $(12+0^2)/4 < 4$; (β) $s=-2$ ($a=-2s$) if $t \neq 0$; (γ) $t=-2$ ($a=-2t$) if $s \neq 0$; (δ) the bracketed form, by (15₁); (ϵ) $t=0$ since $4 \nmid 0-R$; (ζ) $t = \pm 2$ since $4 \nmid 2-R$. There remain 31 reduced forms, and all of them happen to be also E -reduced.

Example 2. To find the reduced forms in the genus of $(7, 7, 7, 1, 1, 1)$, of determinant 324. Here $\Omega=6$, $\Delta=9$, and the reciprocal forms are i.p., so that $C_1=C/12$ is an integer (and also $B/12$); the numbers

represented by the genus of f are the positive integers not of the forms $3^{2q}(3n+2)$ ($q \geq 0$), $3(3n+2)$, or $4n+1, 2$; while C_1 in the reciprocal genus cannot be of the forms $3(3n \pm 1)$, $3n+2$, or $3^{2q+1}(3n+2)$ ($q \geq 1$). Since $a^2 \leq 2d$, $a \leq 8$. By §7 we have only the possibilities $(a, C) = (3, 12)$, $(4, 12)$, $(7, 48)$. If $a=3$ we solve $-972 \equiv R^2 \pmod{12}$, whence $6 \mid R$, and we form $972+0^2=12 \cdot 81$ (discarded since $12 \nmid 81$), $972+6^2=12 \cdot 84$, $12+0^2=3 \cdot 4$, $84+0^2=3 \cdot 28$; we thus write down the form $(3, 4, 28-2, 0, 0)$. If $a=4$ we obtain $1296+0^2=12 \cdot 108$, $1296+6^2=12 \cdot 111$ (discarded), $12+0^2=4 \cdot 3$ (discarded since $3 < 4$), $12+2^2=4 \cdot 4$, $108+0^2=4 \cdot 27$, $108=2^2=4 \cdot 28$; and have the forms $(4, 4, 27, 0, 0, -2)$, $(4, 4, 28, 1, 2, 2)$; but discard the latter as i.p. If $a=7$, $2268+6^2=48 \cdot 48$, $2268+18^2=48 \cdot 54$ (discarded), $48+1^2=7 \cdot 7$; and we have $(7, 7, 7, 1, 1, 1)$. Thus there are three reduced forms in this genus: $(3, 4, 28, -2, 0, 0)$, $(4, 4, 27, 0, 0, -2)$, and $(7, 7, 7, 1, 1, 1)$. As a check, the mass of the genus should be $1/4+1/12+1/6=1/2$, and this agrees with the Eisenstein-Smith [4] formula.

10. Some remarks about minima. The invariance of (19) in §3 proves that among all forms within a class, (19) holds if the coefficients satisfy the inequalities (2) and (9)–(18). An examination of the last part of §3 allows us to drop some of these inequalities, and still assure the validity of (19).

Examples of reduced forms, and of E -reduced forms, in which C is not the minimum of the adjoint form, are easily found; C is, in a reduced form, only the minimum simultaneous with the minimum a .

In an E -reduced form, b is (if it exceeds a) the least number properly represented except for a ; the corresponding number for a reduced form is readily deduced from the list of fourteen forms in §5. Conversely, if the least number represented by F simultaneously with the minimum a by f is desired for an E -reduced form, it can be deduced by inverting the transformations of §5 to obtain the corresponding reduced form.

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ON A GENERALIZED GREEN'S FUNCTION AND CERTAIN OF ITS APPLICATIONS¹

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1. **Introduction.** A theorem of Blaschke in the theory of a.f. 1 c.v. (analytic functions of one complex variable) states: $\sum_{\nu=1}^{\infty} \log |a_{\nu}| > -\infty$, $|a_{\nu}| < 1$, is a necessary and sufficient condition for the existence of a non-negative, harmonic function² $H(z)$, $z \in [\mathfrak{E}^2 - \mathcal{S}_{\nu=1}^{\infty} \{a_{\nu}\}]$, $\mathfrak{E}^2 = \mathcal{E}[|z| < 1]$, which possesses the property that $[H(z) + \log |z - a_{\nu}|]$, $\nu = 1, 2, \dots$, is regular in a neighborhood of $z = a_{\nu}$. By

$$\exp [-H(z) - iB(z)] = f(z)$$

where $B(z)$ is a function conjugate to $H(z)$ we obtain a function $f(z)$, $|f(z)| \leq 1$, $z \in \mathfrak{E}^2$, which possesses factors $(z - a_{\nu})$, $\nu = 1, 2, \dots$.

If one wishes to obtain an analogous result in the theory of a.f. 2 c.v. one must bear in mind at first the following fact:

If $f(z)/g(z)$ is regular in $\mathfrak{E}^2$ we call $g(z)$ a zero function of f in $\mathfrak{E}^2$.

Since every function f regular in $\mathfrak{E}^2$ can be represented in the form $f(z) = \prod (z - a_{\nu}) k_{\nu}(z)$ where $k_{\nu}(z)$ are regular and nonvanishing in $\mathfrak{E}^2$, we need to consider in the theory of a.f. 1 c.v. only *linear zero functions*. In the case of a.f. 2 c.v. we cannot in general represent even polynomials as products of linear functions; therefore, one must use for *zero functions* not only linear expressions but also arbitrary a.f. 2 c.v. [1, p. 1189].³

Furthermore there is lacking in the theory of a.f. 2 c.v. a theorem analogous to the theorem of Riemann, stating that every simply connected domain possessing at least two boundary points can be transformed conformally into $\mathfrak{E}^2$. We cannot therefore limit ourselves to

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² We designate by capital and small letters, respectively, real and complex functions of z_k , $z_k = x_k + iy_k$, and manifolds by English letters, where the upper index denotes the dimension of the manifold. We omit this index for four-dimensional manifolds. We denote by $\mathcal{E}[\dots]$ the set of points whose coordinates satisfy the relations indicated in brackets. $\mathcal{S}$ means the logical sum. A horizontal bar above a letter indicates the closure of the set denoted by the letter.

³ The numbers in brackets refer to the following papers: Stefan Bergman, 1. Proceedings, Akademie van Wetenschappen, Amsterdam, vol. 34 (1932), pp. 1188-1194, 2. Mathematische Annalen, vol. 102 (1934), pp. 324-348, 3. Compositio Mathematica, vol. 3 (1936), pp. 136-173, 4. Compositio Mathematica, vol. 6 (1939), pp. 305-335, 5. Stefan Bergman and Marcinkiewicz, Fundamenta Mathematicae, vol. 33 (1939), pp. 75-94, 6. G. Buler, Bulletin de l'Institut Mathématique de Tomsk, vol. 2 (1939), pp. 164-186, 7. S. Saks, *Theory of the Integral*, Monografie Matematyczne, vol. 7, Warsaw and Lwów, 1937.

consideration of one domain as in the theory of a.f. 1 c.v., but we must study our problem for an arbitrary domain.

In attempting to generalize to the theory of a.f. 2 c.v. the methods which are bases for the proofs of the theorem of Blaschke and other theorems in the theory of meromorphic functions, until now we have limited ourselves to the consideration of a special class of domains (domains with distinguished boundary surface) [2, §1, §3; 3, p. 138]; furthermore it was necessary to introduce instead of harmonic functions an "extended class of functions" which contains biharmonic functions (that is, real or imaginary part of a.f. 2 c.v.). This class possesses various properties analogous to those of harmonic functions, but depends upon the domain [4, p. 306 and p. 319]. In the case of a bicylinder $\mathfrak{E} = \mathcal{E}[|z_k| < r_k, k=1, 2]$, the double harmonic functions, that is, functions H satisfying $\partial^2 H / \partial x_k^2 + \partial^2 H / \partial y_k^2 = 0, k=1, 2$, form the extended class of functions.

2. Functions of the extended class. In the present paper we shall consider a special class of domains, $\mathfrak{M}$, with distinguished boundary surface. Let $h(z_2, \lambda), 0 \leq \lambda \leq 2\pi, h(z_2, 0) = h(z_2, 2\pi)$, be a continuously differentiable function of z_2, λ , which is for every constant λ an analytic function of $z_2, |z_2| \leq 1$, such that:

a°. we have

$$(2.1) \quad \min |h(z_2, \lambda)| \geq \Delta > 0, \quad M^{-1} \leq |\partial h / \partial \lambda| \leq M < \infty;$$

b°. it is possible to represent the curve $b^1(z_2) = \bigcup_{0 \leq \lambda \leq 2\pi} \mathcal{E}[z_1 = h(Z_2, \lambda), Z_2 = z_2], |z_2| \leq 1$, in polar coordinates ρ, ω in the form $\rho = \rho(\omega; z_2)$ where ρ is one-valued, and with the further assumption that

$$(2.2) \quad |\partial \rho / \partial \omega| \leq M, \quad H = 1, 2, \quad M^{-1} \leq |\partial \omega / \partial \lambda| \leq M < \infty.$$

Let $\mathfrak{B}^2(z_2)$ be the region in the plane $Z_2 = z_2$ bounded by $b^1(z_2)$. (See Figure 1, p. 657.) Thus $\mathfrak{B}^2(z_2) = \bigcup_{0 \leq \lambda \leq 2\pi} \bigcup_{0 < \lambda < 2\pi} \mathcal{E}[z_1 = sh(Z_2, \lambda), Z_2 = z_2]$. The domain $\mathfrak{M}$ is now defined as $\mathfrak{M} = \bigcup_{|z_2| < 1} \mathfrak{B}^2(z_2)$. The boundary of $\mathfrak{M}$ consists of two segments of analytic hypersurfaces

$$(2.3) \quad \begin{aligned} j_1^3 &= \bigcup_{0 \leq \lambda \leq 2\pi} \mathfrak{B}^2(\lambda), & \mathfrak{B}^2(\lambda) &= \mathcal{E}[z_1 = h(z_2, \lambda), |z_2| \leq 1], \\ j_2^3 &= \bigcup_{0 \leq \phi_2 \leq 2\pi} \mathfrak{B}^2[\exp(i\phi_2)]. \end{aligned}$$

The intersection

$$\mathfrak{B}^2 = j_1^3 \cdot j_2^3 = \bigcup_{0 \leq \phi_2 \leq 2\pi} \mathcal{E}^1[\exp(i\phi_2)] = \bigcup_{0 \leq \lambda \leq 2\pi} \bigcup_{0 \leq \phi_2 \leq 2\pi} \mathcal{E}[z_1 = h(\exp(i\phi_2), \lambda), z_2 = \exp(i\phi_2)] = \bigcup_{0 \leq \lambda \leq 2\pi} i^1(\lambda),$$

where $i^1(\lambda) = \bigcup_{0 \leq \phi_2 \leq 2\pi} \mathcal{E}[z_1 = h(\exp(i\phi_2), \lambda), z_2 = \exp(i\phi_2)]$, forms the distinguished boundary surface of $\mathfrak{M}$.

Let a real function $G(\phi_2, \lambda)$, $0 \leq \phi_2 \leq 2\pi$, $0 \leq \lambda \leq 2\pi$, be given such that, except for a finite number of values of λ , $G(\phi_2, \lambda)$ is continuous in ϕ_2 and λ for every λ , in the range $0 \leq \phi_2 \leq 2\pi$, except for a finite number of values of ϕ_2 (which can vary with λ) and such that⁴ $\iint |G(\phi_2, \lambda)| d\lambda d\phi_2 = c < \infty$.

The function $D(z_1, z_2) \equiv D(z_1, z_2; G; \mathfrak{M})$ of "extended class $E'(\mathfrak{M})$ " is defined in $\mathfrak{M}$ in the following way:

1°. In every point (z_1, z_2) of $\mathfrak{B}^2(\lambda)$ we define the value of D in such a manner that $D[h(z_2, \lambda), z_2]$ becomes a harmonic function of x_2, y_2 for $|z_2| < 1$, which assumes for $|z_2| = 1$ the values $G(\phi_2, \lambda)$;

2°. In every $\mathfrak{B}^2(z_2)$ we define the values of D in such a manner that $D(z_1, z_2)$ becomes a harmonic function of x_1, y_1 in $\mathfrak{B}^2(z_2)$ which assumes on $b^1(z_2)$ the values $D[h(z_2, \lambda), z_2]$ defined by 1°. We get for D the integral representation

$$(2.4) \quad D(Z_1, Z_2; G; \mathfrak{M}) = \iint G(\phi_2, \lambda) P_1 P_2 d\phi_2 d\chi(\lambda; Z_2),$$

$$P_1 = \frac{1}{2\pi} \frac{1 - |w(Z_1, Z_2)|^2}{1 + |w(Z_1, Z_2)|^2 - 2|w(Z_1, Z_2)| \cos[\chi(\lambda; Z_2) - \arccos w(Z_1, Z_2)]},$$

$$P_2 = \frac{1}{2\pi} \frac{1 - R_2^2}{1 + R_2^2 - 2R_2 \cos(\Phi_2 - \phi_2)},$$

$$R_2 e^{i\Phi_2} = Z_2,$$

where $w(\zeta, Z_2)$, $w(0, Z_2) = 0$, $w'(0, Z_2) > 0$, is the function which maps $\mathfrak{B}^2(Z_2)$ on the unit circle; $\chi(\lambda; Z_2) = \arccos w[h(Z_2, \lambda), Z_2]$ [4, p. 319]. $H(z_1, z_2)$ is said to be $H \in E(\mathfrak{M})$ if $H(z_1, z_2) = \lim_{n \rightarrow \infty} D_n^{(1)}(z_1, z_2) - \lim_{n \rightarrow \infty} D_n^{(2)}(z_1, z_2)$ where $D_n^{(k)}(z_1, z_2) = D(z_1, z_2; G_n^{(k)}; \mathfrak{M})$ is a set of non-decreasing positive functions described above converging uniformly to a limit function in every closed subregion of $\mathfrak{M}$ and almost everywhere on every $S_{0 \leq \phi_2 \leq 2\pi} b^1(re^{i\phi_2})$, $r < 1$. We have then by a classical theorem of Lebesgue [7, p. 28]

$$(2.5) \quad H(0, 0) = \frac{1}{4\pi^2} \iint H[he^{i\phi_2}, \lambda, re^{i\phi_2}] d\phi_2 d\chi(\lambda; 0).$$

The function $L(z_1, z_2)$ is said to be $L \in l(\mathfrak{M})$ if L can be represented in the form

$$(2.6) \quad L(z_1, z_2) = \sum_{n=1}^{\infty} [-\log |g_n(z_1, z_2)| + H_n(z_1, z_2)],$$

⁴ Here and in the following $\iint$ means $\int_{\phi_2=0}^{2\pi} \int_{\lambda=0}^{2\pi}$.

the sum converging uniformly in any closed subregion of $[\mathfrak{M} - \mathfrak{S}_{r=1}^{\infty} \mathfrak{G}_r^2]$, $\mathfrak{G}_r^2 = \mathcal{E}[g_r(z_1, z_2) = 0]$ and converging almost everywhere on every $\mathfrak{S}_{0 \leq \phi_2 \leq 2\pi} \mathfrak{b}^1(re^{i\phi_2})$, $r < 1$, and such that

$$(2.7) \quad \iint L[h(re^{i\phi_2}, \lambda), re^{i\phi_2}] d\phi_2 d\chi(\lambda; 0) = c_r \geq -c_L > -\infty, \quad r < 1;$$

$$(2.8) \quad \iint |L[h(re^{i\phi_2}, \lambda), re^{i\phi_2}]| d\phi_2 d\chi(\lambda, 0) < \infty, \quad r < 1.$$

$g_r(z_1, z_2)$ are a.f. 2 c.v. regular in $\overline{\mathfrak{M}}$ and $H_r(z_1, z_2) \in E(\mathfrak{M})$.

In an analogous manner we introduce the classes $E'(\mathfrak{M}_r)$, $E(\mathfrak{M}_r)$, $I(\mathfrak{M}_r)$, $\mathfrak{M}_r = \mathfrak{S}_{|z_2| < r} \mathfrak{B}^2(z_2) = \mathcal{E}[|z_2| < r, (z_1, z_2) \in \mathfrak{B}^2(z_2)]$.

REMARK 1. Since $\mathcal{E}[z_1 = h(z_2, \lambda), |z_2| < r] \subset \mathfrak{S}^2(\lambda)$ [cf. (2.3)] every function $H(z_1, z_2)$ which is of class $E'(\mathfrak{M})$ or $E(\mathfrak{M})$ belongs to $E'(\mathfrak{M}_r)$ or $E(\mathfrak{M}_r)$ respectively.

3. **Lemmas.** For the proof of the main theorem we need several lemmas.

I. *There exists a function $\alpha_r(Z_1, Z_2)$, $\alpha_r(Z_1, Z_2) < \infty$ for $(Z_1, Z_2) \in \mathfrak{M}_r$, which depends only on $\mathfrak{M}_r$ such that for every non-negative $H(Z_1, Z_2) \in E(\mathfrak{M}_r)$ we have*

$$(3.1) \quad H(Z_1, Z_2) \leq \alpha_r(Z_1, Z_2) H(0, 0).$$

PROOF. According to (2.4) we have for every non-negative $D_{\mathfrak{N}}(Z_1, Z_2) \in E'(\mathfrak{M}_r)$

$$(3.2) \quad D(Z_1, Z_2) \leq \frac{r + R_2}{r - R_2} \frac{1 + |w(Z_1; Z_2)|}{1 - |w(Z_1; Z_2)|} \iint G(\phi_2, \lambda) d\phi_2 d\chi(\lambda; Z_2).$$

Since $\mathfrak{b}^1(Z_2)$ satisfies the conditions a° and b° for every Z_2 , we obtain by Lemma 7 of [5]

$$(3.3) \quad \begin{aligned} M_1^{-1} d \operatorname{arc} w[h(0, \lambda), 0] &\leq d \operatorname{arc} w[h(Z_2, \lambda), Z_2] \equiv d\chi(\lambda, Z_2) \\ &\leq M_1 d \operatorname{arc} w[h(0, \lambda), 0] \equiv M_1 d\chi(\lambda; 0), \end{aligned}$$

M_1 being an appropriate constant. By Lemma 5 of [5],⁵ (3.2) and (2.4) we obtain (3.1) with H replaced by D and with $\alpha_r(Z_1, Z_2) = (r + R_2)(r - R_2)^{-1} [1 + |w(Z_1, Z_2)|] [\delta(Z_1, Z_2)]^{-1}$, $\delta(Z_1, Z_2)$ being an appropriate constant. Since we can approximate $H \in E(\mathfrak{M}_r)$ uniformly in any closed subregion of $\mathfrak{M}_r$ by an appropriate $D(Z_1, Z_2) = D_{\mathfrak{N}}^{(1)}(Z_1, Z_2) - D_{\mathfrak{N}}^{(2)}(Z_1, Z_2)$, $D_{\mathfrak{N}}^{(k)}(Z_1, Z_2) \in E'(\mathfrak{M}_r)$, (3.1) is valid for $H(Z_1, Z_2)$.

⁵ In the Lemma 5 of [5] $\Phi(z)$ is assumed to vanish at $z=0$.

$$(3.4) \quad \Gamma(z_1, z_2; g; \mathfrak{M}_r) = -\log |g(z_1, z_2)| + D(z_1, z_2; \log |g|; \mathfrak{M}_r),$$

$$g \equiv g[h(re^{i\phi_2}, \lambda), re^{i\phi_2}],$$

is designated *Green's function of the extended class corresponding to g and $\mathfrak{M}_r$, $\mathfrak{M}_1 = \mathfrak{M}$* . Here g is supposed to be regular in $\overline{\mathfrak{M}}$.

In every point $(z_1, z_2) \in \mathfrak{F}_r^2 = \mathfrak{S}_{0 \leq \phi_1 \leq 2\pi} \mathfrak{b}^1(re^{i\phi_2})$ where $g(z_1, z_2) \neq 0$, $\Gamma(z_1, z_2; g; \mathfrak{M}_r) = 0$.

Since the set of points of $\mathfrak{F}_r^2$ where $g(z_1, z_2) = 0$ has zero measure, we have

$$(3.5) \quad \iint \Gamma[h(re^{i\phi_2}, \lambda), re^{i\phi_2}; g; \mathfrak{M}_r] d\phi_2 d\chi(\lambda; 0) = 0.$$

In order to avoid tedious operations in our later considerations we suppose in the following that g and g_r have no factors of the form $[z_1 - h(z_2, \lambda)]$, λ const.

II. For $(z_1, z_2) \in \mathfrak{M} + i\mathfrak{I}_1^3$,

$$(3.6) \quad \Gamma(z_1, z_2; g; \mathfrak{M}) \geq 0.$$

PROOF. Since in every $\mathfrak{B}^2(z_2^0)$, $|z_2^0| < 1$, $\Gamma(z_1, z_2; g; \mathfrak{M})$ is a non-negative harmonic function of x_1, y_1 , which may become infinite at a finite number of points a like $-p \log |z_1 - a|$, p positive integer, it is necessary to show only that $\Gamma(z_1, z_2; g; \mathfrak{M}) \geq 0$ for $(z_1, z_2) \in i\mathfrak{I}_1^3 = \mathfrak{S}_{|z_1| < 1} \mathfrak{b}^1(z_2) = \mathfrak{S}_{0 \leq \lambda \leq 2\pi} \mathfrak{F}^2(\lambda)$. (The other possibility is that g has a factor $(z_1 - a)$, but then $\Gamma(z_1, a; g; \mathfrak{M}) = +\infty$.) In every $\mathfrak{F}^2(\lambda)$ we have

$$(3.7) \quad \begin{aligned} \Gamma(z_1, z_2; g; \mathfrak{M}) &= \Gamma[h(z_2, \lambda), z_2; g; \mathfrak{M}] \\ &= -\log |g[h(z_2, \lambda), z_2]| + D[h(z_2, \lambda), z_2; \log |g|; \mathfrak{M}]. \end{aligned}$$

$\Gamma[h(z_2, \lambda), z_2; g; \mathfrak{M}]$ is for every λ a harmonic function of x_2, y_2 in $|z_2| < 1$ except at a finite number of points in which it becomes ∞ and which assumes the boundary values 0 on $|z_2| = 1$. The function (3.7) is obviously non-negative in $|z_2| < 1$.

III. The following relation holds:

$$(3.8) \quad \lim_{r \rightarrow 1} \Gamma(0, 0; g; \mathfrak{M}_r) = \Gamma(0, 0; g; \mathfrak{M}).$$

PROOF. In order to prove (3.8) we shall show that

$$(3.9) \quad \lim_{r \rightarrow 1} D(0, 0; \log |g|, \mathfrak{M}_r) = D(0, 0; \log |g|; \mathfrak{M}).$$

Since $g(z_1, z_2)$ is regular on $\mathfrak{F}_1^2 = \mathcal{S}_{0 \leq \phi_1 \leq 2\pi} b^1(e^{i\phi_1})$, $\mathfrak{F}_1^2 = \mathfrak{F}^2$, we can find by Weierstrass' preparation theorem and the Heine-Borel theorem a finite number of domains $\mathfrak{F}_k = \mathcal{E}[|z_H - \zeta_H^{(k)}| \leq \rho, H=1, 2], k=1, \dots, n$, $(\zeta_1^{(k)}, \zeta_2^{(k)}) \in \mathfrak{F}_1^2$, with the following properties:

- 1°. $\mathcal{S}_{k=1}^n \mathfrak{F}_k$ covers the total (four-dimensional) neighborhood of $\mathfrak{F}_1^2$;
- 2°. in every $\mathfrak{F}_k$,

$$g = (z_2 - \zeta_2^{(k)})^{\mu_k} \prod_{H=1}^{p_k} [z_1 - \alpha_H^{(k)}(z_2)] \Psi_k(z_1, z_2)$$

where p_k, μ_k are non-negative integers, Ψ_k is an a.f. 2 c.v. regular and nonvanishing in $\mathfrak{F}_k$ and $\alpha_H^{(k)}(z_2)$, $\alpha_H^{(k)}(\zeta_2^{(k)}) = \zeta_1^{(k)}$ are algebroid⁶ in a certain neighborhood of $z_2 = \zeta_2^{(k)}$.

We can suppose that r_0 is chosen so near to 1 that g has no factors $(z_2 - re^{i\phi_2})$, $r_0 \leq r < 1$. Hence in order to prove (3.9) it is sufficient to prove that

$$(3.10) \quad \iint \log |z_1 - \alpha_H^{(k)}(z_2)| d\phi_2 d\chi(\lambda; 0), \quad z_2 = re^{i\phi_2},$$

is a continuous function of r for every k and H , the integration being taken over $\mathfrak{F}_r^2 \cdot \mathfrak{F}_k = \mathcal{S}_{\phi_1^{(0)} \leq \phi_1 \leq \phi_1^{(0)} + 2\pi} g^1(re^{i\phi_2})$, $g^1(re^{i\phi_2}) = \mathcal{E}[z_1 = h(re^{i\phi_2}, \lambda), z_2 = re^{i\phi_2}, \lambda_1(r, \phi_2) \leq \lambda \leq \lambda_2(r, \phi_2)]$.

Let $\delta > 0$, $\epsilon > 0$ and $\mathfrak{R}_\delta^2 = \mathcal{E}[|z_1 - \zeta_1^{(k)}| \leq \delta, z_2 = re^{i\phi_2}]$. We can determine an r_0 so near to 1 that for $r_0 \leq r \leq 1$ and $\psi - \epsilon \leq \phi_2 \leq \psi + \epsilon$ where $\psi = \arg \zeta_2^{(k)}$, $\alpha_H^{(k)}(re^{i\phi_2})$ lies⁷ in $\mathfrak{R}_\delta^2$ and the point $\alpha_H^{(k)}(re^{i\phi_2})$ is not contained in $g^1(re^{i\phi_2})$ for $r_0 \leq r \leq 1$, $\phi_2^{(1)} \leq \phi_2 \leq \psi - \epsilon$ or $\psi + \epsilon \leq \phi_2 \leq \phi_2^{(2)}$.

We dissect every $g^1(re^{i\phi_2})$, $r_0 \leq r \leq 1$, $\psi - \epsilon \leq \phi_2 \leq \psi + \epsilon$, into two parts; $g_0^1(re^{i\phi_2}) = \mathfrak{R}_\delta^2 \cdot g^1(re^{i\phi_2}) = \mathcal{E}[z_1 = h(z_2, \lambda), z_2 = re^{i\phi_2}, \lambda^{(1)}(\delta, \phi_2) \leq \lambda \leq \lambda^{(2)}(\delta, \phi_2)]$ and $f^1(re^{i\phi_2}) = g^1(re^{i\phi_2}) - g_0^1(re^{i\phi_2})$.

(3.10) Taken over $\mathcal{S}_{\phi_2^{(0)} \leq \phi_2 \leq \phi_2^{(0)} + 2\pi} g^1(re^{i\phi_2})$, $\mathcal{S}_{\psi - \epsilon \leq \phi_2 \leq \psi + \epsilon} f^1(re^{i\phi_2})$ is obviously a continuous function of r , since $\alpha_H^{(k)}(re^{i\phi_2})$ does not lie on these surfaces. By (2.1) and (2.2) there exists by Lemmas 3 and 4 of [5] a $P = P(M, \Delta)$ such that every circle of radius P , tangent to $b^1(re^{i\phi_2})$ at an arbitrary point $h(re^{i\phi_2}, \lambda)$ of $b^1(re^{i\phi_2})$ lies either entirely inside of $\mathfrak{B}^2(re^{i\phi_2})$ or outside of this domain.

⁶ A function $\alpha = \alpha(z)$ satisfying the equation $\alpha^m + g_1(z)\alpha^{m-1} + \dots + g_m(z) = 0$, where $g_p(z)$, $p=1, 2, \dots, m$, are a.f. of 1 c.v. regular in the domain $\mathfrak{U}^2$ is said to be "algebroid in $\mathfrak{U}^2$."

⁷ Here and in the following part of this section we write for the sake of shortness $h(re^{i\phi_2}, \lambda)$ instead of $[h(re^{i\phi_2}, \lambda), re^{i\phi_2}]$ and $\alpha_H^{(k)}(re^{i\phi_2})$ instead of $[\alpha_H^{(k)}(re^{i\phi_2}), re^{i\phi_2}]$. (The points are supposed to be points of the space (z_1, z_2) , lying in the plane $z_2 = re^{i\phi_2}$, but we omit the coordinate z_2 .)

If $|\lambda - \lambda^0| > PM^{-1}$, then we have the two following possibilities:

1°. $h(re^{i\phi_2}, \lambda)$ lies in the triangle $\mathfrak{T}^2$ and we have by (2.2)

$$|\omega - \omega^0| \geq PM^{-2},$$

and from this $|h(re^{i\phi_2}, \lambda) - \alpha_H^{(k)}(re^{i\phi_2})| \geq (\frac{1}{2})^{3/2} \Delta PM^{-2}$, since $h(re^{i\phi_2}, \lambda) \in \mathfrak{T}^2$;

2°. $h(re^{i\phi_2}, \lambda)$ lies outside of $\mathfrak{T}^2$ and then, since $\alpha_H^{(k)}(re^{i\phi_2})$ lies between $h(re^{i\phi_2}, \lambda^0)$ and the center of the circle, we have

$$|h(re^{i\phi_2}, \lambda) - \alpha_H^{(k)}(re^{i\phi_2})| \geq P.$$

Since we can suppose that $M \geq 1$ and $P \leq 1$ we have finally that

$$(3.11) \quad |h(re^{i\phi_2}, \lambda) - \alpha_H^{(k)}(re^{i\phi_2})| \geq (\frac{1}{2})^{3/2} \Delta PM^{-1} \cdot |\lambda - \lambda^0|_{M^{-1}}$$

where $|\lambda - \lambda^0|_{M^{-1}}$ is $|\lambda - \lambda^0|$ if $|\lambda - \lambda^0| \leq M^{-1}$, and is M^{-1} if $|\lambda - \lambda^0| \geq M^{-1}$.

Thus the integral (3.10) taken over $\mathfrak{S}_{\psi-\delta, \psi+\delta, \phi_2}^1(re^{i\phi_2})$ is smaller in absolute value than

$$M \int_{\psi-\delta}^{\psi+\delta} \int_{\lambda^0(\delta, \phi_2)}^{\lambda^2(\delta, \phi_2)} |\log((\frac{1}{2})^{3/2} \Delta PM^{-1} \cdot |\lambda - \lambda^0|_{M^{-1}})| d\lambda d\phi_2$$

which converges to 0 as $\delta \rightarrow 0$.

4. The main theorem. Let $\{g_\nu(z_1, z_2)\}$ designate a set of a.f. 2 c.v. regular in $\mathfrak{M}$ and possessing the property: A. For every $r < 1$ there exists an $N(r)$ so that $g_\nu(z_1, z_2), \nu > N(r)$, does not vanish in $\overline{\mathfrak{M}}_r = \mathfrak{S}_{|z_1| \leq r, \mathfrak{B}^2(z_2)}$. Then

$$(4.1) \quad \sum_{\nu=1}^{\infty} \Gamma(0, 0; g_\nu; \mathfrak{M}) < \infty$$

is a necessary and sufficient condition for the existence of a function $F(z_1, z_2) \in l(\mathfrak{M})$, $F(0, 0) < \infty$, which possesses the property that for every ν $F(z_1, z_2) + \log|g_\nu(z_1, z_2)|$ is regular on $\mathfrak{G}_\nu^2$ except at the intersection points $\mathfrak{G}_\nu^2 \cdot \mathfrak{G}_n^2, n \neq \nu$.

PROOF. (4.1) is necessary. The function F becomes infinite like $-\log|g_\nu(z_1, z_2)|$ on $\mathfrak{G}_\nu^2, \nu = 1, 2, \dots$. Since by A, $\mathfrak{G}_\nu^2$ for $\nu > N(r)$ lies outside of $\mathfrak{M}_r$ and therefore a fortiori outside of $\mathfrak{M}_\rho, \rho \leq r$, $F(z_1, z_2) - \sum_{\nu=1}^{N(r)} \Gamma(z_1, z_2; g_\nu; \mathfrak{M}_\rho)$ has no singularities in $\mathfrak{M}_\rho$. Since $-\log|g_\nu(z_1, z_2)|$ for $\nu > N(r)$ are regular biharmonic functions in $\overline{\mathfrak{M}}_\rho$ we have by Remark 1 that $[-\log|g_\nu(z_1, z_2)| + H_\nu(z_1, z_2)] \in E(\mathfrak{M}_\rho)$ for $\nu > N(r)$ and therefore that

$$\begin{aligned}
 (4.2) \quad & \iint \sum_{r=N(r)+1}^{\infty} [-\log |g_r| + H_r] d\psi_2 d\chi(\lambda; 0) \\
 &= \frac{1}{4\pi^2} \sum_{r=N(r)+1}^{\infty} [-\log |g_r(0, 0)| + H_r(0, 0)].
 \end{aligned}$$

Since

$$\begin{aligned}
 (4.3) \quad & F(z_1, z_2) - \sum_{r=1}^{N(r)} \Gamma(z_1, z_2; g_r; \mathfrak{M}_\rho) \\
 &= \lim_{m \rightarrow \infty} \sum_{r=N(r)+1}^m [-\log |g_r(z_1, z_2)| + H_r(z_1, z_2)] + P(z_1, z_2)
 \end{aligned}$$

is absolutely integrable, where

$$P(z_1, z_2) = \sum_{r=1}^{N(r)} [H_r(z_1, z_2) - D(z_1, z_2; g_r; \mathfrak{M}_\rho)]$$

[cf. (2.6), (2.8), and (3.4)], it follows by (4.2), (4.3) and the Lebesgue theorem [7, p. 29] that³

$$\begin{aligned}
 (4.4) \quad & \iint F d\phi_2 d\chi(\lambda; 0) = \iint \left[F - \sum_{r=1}^{N(r)} \Gamma_{r\rho} \right] d\phi_2 d\chi(\lambda; 0) \\
 &= \iint \left[\sum_{r=N(r)+1}^{\infty} (-\log |g_r| + H_r) + P \right] d\phi_2 d\chi(\lambda; 0) \\
 &= \frac{1}{4\pi^2} \left\{ \sum_{r=N(r)+1}^{\infty} [-\log |g_r(0, 0)| + H_r(0, 0)] + P(0, 0) \right\} \\
 &= \frac{1}{4\pi^2} \left[F(0, 0) - \sum_{r=1}^{N(r)} \Gamma(0, 0; g_r; \mathfrak{M}_\rho) \right].
 \end{aligned}$$

By (2.7), (3.6) the right-hand member is $\geq -C_F$ and therefore for $t \leq N(r)$ we have by II

$$(4.5) \quad \sum_{r=1}^t \Gamma(0, 0; g_r; \mathfrak{M}_\rho) \leq 4\pi^2 C_F + F(0, 0).$$

By Lemma III we can replace in (4.5) $\mathfrak{M}_\rho$ by $\mathfrak{M}$ and since this inequality is true for every t , (4.1) holds.

In order to prove that (4.1) is sufficient we shall show that $\sum_{r=1}^{\infty} \Gamma(z_1, z_2; g_r; \mathfrak{M}) \in I(\mathfrak{M})$. For $r < 1$, $N(r) < m < \infty$,

³ The arguments of g_r , H_r , P , $\Gamma_{r\rho}$ lacking in (4.2) and (4.4) are assumed to be $[h(\rho e^{i\phi_1}, \lambda), \rho e^{i\phi_1}]$.

$\Gamma_{r\rho}$ in (4.4) is $\Gamma[h(\rho e^{i\phi_1}, \lambda), \rho e^{i\phi_1}; g_r; \mathfrak{M}_\rho]$.

$$\sum_{r=N(r)+1}^{\infty} \Gamma(z_1, z_2; g_r; \mathfrak{M}) \in E'(\mathfrak{M}_r).$$

By II and I we have

$$(4.6) \quad \sum_{r=N(r)+1}^{\infty} \Gamma(z_1, z_2; g_r; \mathfrak{M}) \leq \alpha_r(z_1, z_2) \sum_{r=N(r)+1}^{\infty} \Gamma(0, 0; g_r; \mathfrak{M}).$$

Therefore by II and (4.1), $\sum_{r=N(r)+1}^{\infty} \Gamma(z_1, z_2; g_r; \mathfrak{M})$ converges uniformly in any closed subregion of $\mathfrak{M}$.

Since

$$(4.7) \quad \iint \left\{ \sum_{r=N(r)+1}^{\infty} \Gamma[h(re^{i\phi_2}, \lambda), re^{i\phi_2}; g_r; \mathfrak{M}] \right\} d\phi_2 d\chi(\lambda; 0) \\ = \frac{1}{4\pi^2} \sum_{r=N(r)+1}^{\infty} \Gamma(0, 0; g_r; \mathfrak{M}),$$

by II and the known theorem of Lebesgue [7, p. 28]

$$\sum_{r=N(r)+1}^{\infty} \Gamma(z_1, z_2; g_r; \mathfrak{M})$$

exists almost everywhere on $\mathfrak{F}_r^2$, $r < 1$, and we have

$$(4.8) \quad \iint \left\{ \sum_{r=N(r)+1}^{\infty} \Gamma[h(re^{i\phi_2}, \lambda), re^{i\phi_2}; g_r; \mathfrak{M}] \right\} d\phi_2 d\chi(\lambda; 0) \\ = \frac{1}{4\pi^2} \sum_{r=N(r)+1}^{\infty} \Gamma(0, 0; g_r; \mathfrak{M}).$$

Therefore by (4.1) $F(z_1, z_2) = \sum_{r=1}^{\infty} \Gamma(z_1, z_2; g_r; \mathfrak{M})$ exists in $\mathfrak{M} = \lim_{r \rightarrow 1} \mathfrak{M}_r$. It follows by II that $F(z_1, z_2) \geq 0$, $(z_1, z_2) \in \mathfrak{M} + \mathfrak{I}_1^3$, and by (4.8) and (4.1) we have that

$$(4.9) \quad \iint \{F[h(re^{i\phi_2}, \lambda), re^{i\phi_2}]\} d\phi_2 d\chi(\lambda; 0) \geq 0, \quad r < 1.$$

REMARK. Since double harmonic functions are the extended class for any bicylinder, an analogous procedure shows that in the case of a bicylinder $\mathcal{E}[|z_k| < 1, k=1, 2]$ A (see p. 658) can be replaced by the (weaker) hypothesis that $g_r(z_1, z_2)$, $r \geq N(r)$, do not vanish in $\mathcal{E}[|z_k| \leq r, k=1, 2]$. Furthermore we wish to notice that from results of Buler [6] one can obtain in this case certain properties of lines $\mathcal{E}[F(z_1, z_2) = 0, |z_1| = A|z_2|^\alpha, A, \alpha > 0, |z_k| < 1]$ if F is biharmonic in $[\mathcal{E} - \mathcal{S}_{r=1}^\infty \mathfrak{U}_r^2]$.

Finally we wish to indicate that a procedure analogous to those of [4] enables us to associate with every function f , meromorphic in $\mathfrak{M}$, a characteristic function $T(r, f)$, $r < 1$. Using the results of [5] and those of a work of Bers⁹ as well as the theorem of this paper it is possible to show that, under certain hypotheses, $|f|$ possesses boundary values almost everywhere on $\mathfrak{F}^2$, if the $T(r, f)$ is uniformly bounded as $r \rightarrow 1$.

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⁹ The paper of Bers will appear in American Journal of Mathematics. A preliminary report of his work may be found in Comptes Rendus de l'Académie des Sciences, Paris, vol. 208 (1939), pp. 1273-1275 and 1475-1477.

MONOTONIC COLLECTIONS OF PERIPHERALLY SEPARABLE CONNECTED DOMAINS¹

F. B. JONES

In my vain attempts to construct an example of a Moore space which is normal but not metric,² I have discovered a few simple and useful theorems about metric spaces which sound familiar but surprisingly do not seem to be known or in the literature. The following is such a theorem and deals with certain conditions under which a monotonic collection of domains contains a *countable* monotonic subcollection running upward through it. Application of the theorem to certain well ordered sequences is immediate.

Definitions.³ A collection G of point sets is said to be *monotonic* provided that if g_1 and g_2 are elements of G then either g_1 contains g_2 or g_2 contains g_1 . A subcollection H of a collection G of point sets is said to *run upward through* G provided that if g is an element of G there exists an element of H which contains g .

DEFINITION. A point set is said to be *peripherally separable* provided that its boundary is separable.

Let S denote a locally connected metric space.

¹ Presented to the Society, February 22, 1941.

² See F. B. Jones, *Concerning normal and completely normal spaces*, this Bulletin, vol. 43 (1937), pp. 671-677.

³ For the definition of certain terms and phrases, the reader is referred to R. L. Moore's *Foundation of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932, or to W. Sierpiński's *Introduction to General Topology*, Toronto, 1934, translated by C. C. Krieger.

THEOREM A. *If G is a monotonic collection of peripherally separable connected domains of S then some countable monotonic subcollection of G runs upward through G .⁴*

PROOF. Let H denote a well ordered subcollection of G which runs upward through G such that if h_2 of H follows h_1 of H , then h_1 is a proper subset of h_2 . Suppose that H is uncountable. For each element h of H , let β_h denote the boundary of h . Let θ denote the infinite set of real numbers $0, 1, \frac{1}{2}, \frac{1}{3}, \dots$, and for each point x of β_h let r_x denote the largest number of θ such that the circular region with center at x and radius equal to r_x lies in some element of H . For each n , $n=1, 2, 3, \dots, \infty$, let M_{h_n} denote the set of all points x of β_h such that $r_x = 1/n$. Since β_h is separable, every subset of β_h is separable. Hence for each n , $n=1, 2, 3, \dots, \infty$, there exists a countable subset N_{h_n} of M_{h_n} which is dense in M_{h_n} . Since H is uncountable and, for each element h of H , $\sum_{n=1}^{\infty} N_{h_n}$ is countable, there exists a countably infinite sequence $h_1, h_2, h_3, \dots$ of elements of H such that for each positive integer i , h_{i+1} contains h_i together with all points y such that, for some n , the distance from y to N_{h_n} is less than $1/n$. Again since H is uncountable, some element g_1 of H contains $\sum h_i$. Let g_2 denote the first element of H following g_1 in H . Since g_2 contains a point not in g_1 , g_2 contains a boundary point X of $\sum h_i$. Space being locally connected, there exists a sequence of points $x_{1n_1}, x_{2n_2}, x_{3n_3}, \dots$ having X as a sequential limit point such that for each i , $i=1, 2, 3, \dots$, n_i is a positive integer and x_{in_i} belongs to $N_{h_{in_i}}$. Obviously $n_i \rightarrow \infty$ as $i \rightarrow \infty$. But for some positive integer k , every point at a distance less than $1/k$ from X lies in g_2 . Hence there exists an integer $\bar{i}$ such that when $i > \bar{i}$ every point at a distance less than $1/(k+1)$ from x_{in_i} lies in g_2 . But x_{in_i} belongs to $N_{h_{in_i}}$, $i=1, 2, 3, \dots$. Hence when $i > \bar{i}$, $1/n_i \geq 1/(k+1)$, and hence $n_i \leq k+1$. This is a contradiction since, as has already been pointed out, $n_i \rightarrow \infty$ as $i \rightarrow \infty$. So the assumption that H is uncountable is false.

COROLLARY. *In a locally connected metric space, every well ordered*

⁴ Compare with certain of the properties discussed by Sierpiński in his paper, *Sur l'équivalence de trois propriétés des ensembles abstraits*, *Fundamenta Mathematicae*, vol. 2 (1921), pp. 179-188. This paper contains references to the work of Fréchet which is closely related to Theorem A. The relation of Theorem A to certain well known covering theorems (associated with the names of Borel, Lebesgue, and Lindelöf) is evident. See also R. L. Moore, *An acknowledgement*, *Fundamenta Mathematicae*, vol. 8 (1926), pp. 374-375; R. G. Lubben, *Concerning limiting sets in abstract spaces II*, *Transactions of this Society*, vol. 43 (1938), pp. 482-493; and the references therein.

increasing sequence of peripherally separable connected domains is countable.

Examples and remarks. If the hypothesis of the theorem is weakened in any respect and not strengthened in some other respect, the conclusion does not follow. This can be seen by considering the well known space which may be roughly described as composed of uncountably many straight line intervals having one common endpoint and each pair being perpendicular at that point. This example also shows (by removing $\aleph_1$ of the free endpoints one at a time) that if the word *upward* in Theorem A is changed to *downward* (and a natural interpretation given to its meaning), the resulting proposition is false. Furthermore, the theorem does not necessarily hold true for non-metric spaces, even if the space be a Moore space. The only example which I have been able to discover that shows this latter situation is unfortunately too complicated to warrant its inclusion in this paper. In still another direction, if S is metric but not locally connected, the theorem is again false. For consider a space constructed roughly in the following way. (1) Let α denote an uncountable well ordered sequence of distinct points $A_1, A_2, A_3, \dots$ such that no point of α is preceded by uncountably many points of α . (2) For each point A_s of the sequence α , join A_s to A_{s+1} with a unit straight line interval of points such that no two such intervals have a point in common except when the end of one is the beginning of the other and preserve the ordinary limit point relations as given by these intervals (not by α). Let Q denote the space obtained so far. It consists of uncountably many mutually exclusive straight line *rays*. (3) To connect the space, a process involving an uncountable well ordered sequence of additions to Q is performed. For each point A of α having no immediate predecessor in α , select a simple sequence $B_{1A}, B_{2A}, B_{3A}, \dots$ of points of α approaching A in α . For each positive integer i , add to Q a straight line interval T_{iA} which is $\frac{1}{i}$ unit long, which has one end at B_{iA} , and which is perpendicular to each other interval (whether added in (2) or (3)) containing B_{iA} . Let A be the sequential limit point of the end-points of the intervals $T_{1A}, T_{2A}, T_{3A}, \dots$ which are distinct from $B_{1A}, B_{2A}, B_{3A}, \dots$ respectively. (4) The sum of all the intervals thus put together constitutes a metric space S . For each point A_s of α , let D_s denote the sum (except for possibly the point A_s itself) of all the intervals in S containing a point of α which precedes A_s in α . The sequence $D_1, D_2, D_3, \dots$ is a monotonic collection of connected domains each of which has only one boundary point. Nevertheless no countable subsequence of $D_1, D_2, D_3, \dots$ runs through it.

In view of the fact that the components of a domain in a locally connected space are themselves domains, one might suspect the following to be true: In a *connected* locally connected metric space every monotonic collection of peripherally separable domains contains a countable subcollection running upward through it. This is false as can be seen from the example of a space composed of uncountably many perpendicular intervals described above. However, the following proposition is true: *In a metric space, every monotonic collection of separable domains contains a countable subcollection running upward through it.* This follows from well known results.⁵

Applications. The application of Theorem A to the problem mentioned in the opening paragraph of this paper is more or less evident. It can also be used to establish rather easily the following known result: *A connected locally connected, locally peripherally separable, metric space is completely (perfectly) separable.*⁶ The proof is direct and almost immediate.

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⁵ See pages 300 and 301 of Alexandroff's paper, *Über die metrisation der im Kleinen kompakten topologischen Räume*, Mathematische Annalen, vol. 92 (1924), pp. 294-301, in particular.

⁶ F. B. Jones, *A theorem concerning locally peripherally separable spaces*, this Bulletin, vol. 41 (1935), pp. 437-439.

ON THE MAPPING OF THE SETS OF 24 POINTS OF THE SYMMETRIC SUBSTITUTION GROUP G_{24} IN ORDINARY SPACE UPON A HYPERQUADRIC CONE

EARL WALDEN

Introduction. The mapping of the sextuples of the symmetric substitution group G_6 in a plane upon a quadric has been done by Emch.¹ The 24 permutations of 4 elements x_1, x_2, x_3, x_4 considered as projective coordinates in ordinary space determine a configuration² which may be mapped on a hypersurface in S_4 . I shall show that the hypersurface on which we will map is a hyperquadric cone. The map of every configuration on the hyperquadric will be a configuration in ordinary space, invariant under the G_{24} .

The mapping of the G_{24} . We shall represent the elementary symmetric functions as follows:

$$\phi_1 = x_1 + x_2 + x_3 + x_4,$$

$$\phi_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4,$$

$$\phi_3 = x_1x_2x_3 + x_1x_3x_4 + x_1x_2x_4 + x_2x_3x_4,$$

$$\phi_4 = x_1x_2x_3x_4.$$

Let $y_i = A_i\phi_1^4 + B_i\phi_1^3\phi_2 + C_i\phi_1^2\phi_2^2 + D_i\phi_1\phi_3 + E_i\phi_4$ where $i=1, 2, 3, 4, 5$. There are five linearly independent y 's. We shall consider the y 's as the coordinates of a point in S_4 . Thus to each point in (x) , and consequently to each of 24 points in (x) , corresponds a point (y) in S_4 . The locus of the points (y) is a hypersurface of some order in S_4 .

Let us choose five linearly independent y 's. (For every choice of y 's we will get some hypersurface and all these hypersurfaces will be linearly related.)

$$\rho y_1 = \sum x_1^4 = \phi_1^4 - 4\phi_1^2\phi_2 + 2\phi_2^2 + 4\phi_1\phi_3 - 4\phi_4,$$

$$\rho y_2 = \sum x_1^3x_2 = \phi_2^2 - 2\phi_1\phi_3 + 2\phi_4,$$

$$\rho y_3 = \sum x_1^3x_3 = \phi_1^2\phi_2 - 2\phi_2^2 - \phi_1\phi_3 + 4\phi_4,$$

$$\rho y_4 = \sum x_1^3x_2x_3 = \phi_1\phi_3 - 4\phi_4,$$

$$\rho y_5 = \sum x_1x_2x_3x_4 = \phi_4.$$

If we eliminate the ϕ 's we get a hyperquadric cone Q given by

¹ This Bulletin, vol. 33 (1927), pp. 745-750.

² Veronese Annali di Matematica, (2), vol. 2, p. 93.

$$(1) \quad y_1(y_2 + 2y_4 + 6y_5) + 2y_2 - y_3^2 - y_4^2 + 4y_2y_4 + 12y_2y_5 - 2y_3y_4 = 0.$$

The rank of the matrix of this hyperquadric cone is three. This means that the hyperquadric has a line of vertices. The partial derivatives,

$$\begin{aligned} \frac{\partial Q}{\partial y_1} &= y_2 + 2y_4 + 6y_5, & \frac{\partial Q}{\partial y_2} &= y_1 + 4y_2 + 4y_4 + 12y_5, \\ \frac{\partial Q}{\partial y_3} &= -2y_3 - 2y_4, & \frac{\partial Q}{\partial y_4} &= 2y_1 + 4y_2 - 2y_3 - 2y_4, \\ \frac{\partial Q}{\partial y_5} &= 6y_1 + 12y_2 \end{aligned}$$

all vanish at the points $V(-4, 2, 4, -4, 1)$ and $V'(4, -2, -1, 1, 0)$ and any point on the join of these two points. Hence this join VV' is the vertex of the hyperquadric cone.

Next, the exceptional points of the $(1, 24)$ transformation will be considered. To the intersections of $\phi_1=0$, $\phi_2=0$, $\phi_4=0$, that is, $(1, \omega, \omega^2, 0)$, $(1, \omega, 0, \omega^2)$, $(1, 0, \omega, \omega^2)$, $(0, 1, \omega, \omega^2)$, $(1, \omega^2, \omega, 0)$, $(1, \omega^2, 0, \omega)$, $(1, 0, \omega^2, \omega)$, $(0, 1, \omega^2, \omega)$, corresponds $y_1=y_2=y_3=y_4=y_5=0$, which represents no point. These 8 points are fundamental points of the transformation. Hereafter they will be called the F -points.

To the first neighborhood of the F -points corresponds the join of $V'(4, -2, -1, 1, 0)$ and $V(-4, 2, 4, -4, 1)$. For example, to the first neighborhood of $(1, \omega, \omega^2, 0)$, that is, $P_d(1+d_1, \omega+d_2, \omega^2+d_3, d_4)$, corresponds

$$\begin{aligned} y_1 &= 4(d_1 + d_2 + d_3) = 4(d_1 + d_2 + d_3 + d_4) - 4(d_4), \\ y_2 &= -2(d_1 + d_2 + d_3) = -2(d_1 + d_2 + d_3 + d_4) + 2(d_4), \\ y_3 &= (d_1 + d_2 + d_3 - 3d_4) = -1(d_1 + d_2 + d_3 + d_4) + 4(d_4), \\ y_4 &= d_1 + d_2 + d_3 - 3d_4 = 1(d_1 + d_2 + d_3 + d_4) - 4(d_4), \\ y_5 &= d_4 = 0(d_1 + d_2 + d_3 + d_4) + 1(d_4), \end{aligned}$$

which is the join of V and V' . If P_d is on $\phi_1=0$, to it corresponds the point $V(-4, 2, 4, -4, 1)$. To any point on $\phi_1=0$, $\phi_4=0$ corresponds the point $T(2, 1, -2, 0, 0)$. A generic hyperplane, $y_1+\lambda_1y_2+\lambda_2y_3+\lambda_3y_4+\lambda_4y_5=0$, cuts Q in a quadric q and the line VT in a point R on q to which corresponds in (x) a quartic surface

$$(2) \quad \begin{aligned} \phi_1^4 + (\lambda_2 - 4)\phi_1^2\phi_2 + (4 - 2\lambda_1 - \lambda_2 + \lambda_3)\phi_1\phi_3 + (2 + \lambda_1 - 2\lambda_2)\phi_2^2 \\ + (2\lambda_1 + 4\lambda_2 - 4\lambda_3 + \lambda_4 - 4)\phi_4 = 0, \end{aligned}$$

which has $\phi_1=0, \phi_4=0$ (which is composed of 4 lines) as double tangents. That is, the line $\phi_1=0, x_4=0$ is tangent to (2) at the points $(1, \omega, \omega^2, 0)$ and $(1, \omega^2, \omega, 0)$; the line $\phi_1=0, x_3=0$ is tangent to (2) at the points $(1, \omega, 0, \omega^2)$ and $(1, \omega^2, 0, \omega)$; the line $\phi_1=0, x_2=0$ is tangent to (2) at the points $(1, 0, \omega, \omega^2)$ and $(1, 0, \omega^2, \omega)$ and the line $\phi_1=0, x_1=0$ is tangent to (2) at $(0, 1, \omega, \omega^2)$ and $(0, 1, \omega^2, \omega)$. Thus to a generic point R on VT corresponds the first neighborhood of the F points, on $\phi_1=0, \phi_4=0$.

To a hyperplane through VV'

$$y_1 + \lambda_1 y_2 + \lambda_2 y_3 + (2\lambda_1 + \lambda_2 - 4)y_4 + (6\lambda_1 - 12)y_5 = 0$$

corresponds the quartic

$$\phi_1^4 + (\lambda_2 - 4)\phi_1^2\phi_2^2 + (2 + \lambda_1 - 2\lambda_2)\phi_2^2 = 0,$$

which is the product of two quadrics of the form $\phi_1^2 + \mu\phi_2 = 0$. Thus a generic hyperplane of the bundle through VV' cuts Q in two planes to which correspond in (x) two quadrics of the symmetric pencil $\phi_1^2 + \mu\phi_2 = 0$.

To a hyperplane through VV' tangent to Q at some point $P(a, b, c, d, e)$ on Q and not on VV' ,

$$(b + 2d + be)y_1 + (a + 4b + 4d + 12e)y_2 - (2c + 2d)y_3 \\ + (2a + 4b - 2c - 2d)y_4 + (6a + 12b)y_5 = 0,$$

corresponds

$$(b + 2d + be)\phi_1^4 - (4b + 10d + 2c + 24e)\phi_1^2\phi_2^2 \\ + (a + 6b + 4c + 12d + 24e)\phi_2^2 = 0.$$

This quartic surface is the square of a quadric if $(4b + 10d + 2c + 24e)^2 - 4(b + 2d + be)(a + 6b + 4c + 12d + 24e) = 0$ or if $-4[(2b^2 - c^2 - d^2) + a(b + 2d + 6e) + 4bd + 12be + 2cd] = 0$. But this is simply the condition that the point $P(a, b, c, d, e)$ lie on Q which we assumed in the beginning. Thus to a hyperplane through VV' tangent to Q at some point P not on VV' corresponds a quartic which is the square of a quadric $\phi_1^2 + \mu\phi_2 = 0$.

To a hyperplane through $VV'T$,

$$y_1 + 6y_2 + 4y_3 + 12y_4 + 24y_5 = 0,$$

corresponds the quartic $\phi_1^4 = 0$ which is the plane $\phi_1 = 0$ counted four times.

To a hyperplane through the line VT ,

$$y_1 + (2\lambda_2 - 2)y_2 + \lambda_2 y_3 + \lambda_2 y_4 + (4\lambda_2 - 8\lambda_2 + 8)y_5 = 0,$$

corresponds the quartic

$$\phi_1^4 + (\lambda_2 - 4)\phi_1^2\phi_2 + (8 - 5\lambda_2 + \lambda_3)\phi_1\phi_3 = 0,$$

which is composed of the plane $\phi_1=0$ and the cubic surface $\phi_1^3 + (\lambda_2 - 4)\phi_1\phi_2 + (8 - 5\lambda_2 + \lambda_3)\phi_3 = 0$.

In general if a hypersurface contains a plane of Q , a factor $\phi_1^2 + \mu\phi_2$ splits off of the corresponding surface in (x) . And if the hypersurface contains the line VT , the factor ϕ_1 splits off in (x) .

Mapping of intersections of the hyperquadric cone. A generic hypersurface H_n cuts Q in a surface F_{2n} to which corresponds in (x) a surface F'_{4n} . From the form of the transformation one can see that each of the four lines $\phi_1=0$ and $\phi_4=0$ is an n -fold double tangent, and each of the 6 points of intersection of $\phi_1=0$, $\phi_2=0$, $\phi_3=0$, $(1, i, -1, i)$, $(1, -i, -1, i)$, $(1, i, -i, -1)$, $(1, -i, i, -1)$, $(i, -i, 1, -1)$, $(i, -i, -1, 1)$, is an n -fold point of F'_{4n} .

To a generic surface F'_n in (x) corresponds on Q a surface whose order can always be determined. Suppose F'_n does not pass through the F points. The equation of F'_n will contain a term of the form ϕ_3^m where $3m=n$. A generic quartic surface F'_4 and another quartic surface f'_4 cuts F'_n , or F'_{3m} , in $48m$ points which form $2m$ sets of 24 points each. To these $2m$ points correspond in (y) the $2m$ points that are on the plane of intersection of the two hyperplanes F and f that correspond to the two quartic surfaces F'_4 and f'_4 . But these $2m$ points are the intersections of the surface in (y) , that corresponds to F'_n , and the plane common to F and f . Thus the surface in (y) that corresponds to F'_n is of order $2m$, where $3m=n$.

The surface F_{2n} on Q is cut out by a hypersurface H which may pass through a plane of Q . For example, when F'_n is a sextic surface, H is a hyperquadric, call it H_2 , which passes through a plane of Q . That is, the intersection of H_2 and Q is composed of a plane and a cubic to which corresponds in (x) a quadric and a cubic surface. More generally H_m cuts Q in a surface to which corresponds in (x) a surface of order $4m$. In order that it reduce to $3m$ it is necessary that a factor of order m split off. We have seen that the factors will be of the form ϕ_1^d and $(\phi_1^2 + \mu\phi_2)^\beta$, where $d + 2\beta = m$, and H_m will contain VT two times and β planes of Q . For example, if $n=9$ and $m=3$, H_m will be a cubic that contains VT and one plane of Q .

Suppose two hypersurfaces H_m and H_n cut Q . To this intersection C will correspond in (x) the intersections of two surfaces F'_{4m} and F'_{4n}

which is a curve C' of order $16mn$. Thus to C_{mn} in (y) correspond in (x) C'_{16mn} .

To a generic curve C'_n in (x) which is the complete intersection of two symmetric surfaces F'_r and F'_s , where $rs=n$, corresponds a curve in S_4 whose order can be determined. If the two surfaces do not go through the F points, each surface will have a term of the form ϕ_3^d where $3d=r$ or $3d\beta=s$. A surface F'_4 will intersect F'_r and F'_s , and consequently C'_n , in $36d$ points to which corresponds in (y) $36d\beta/24$ points which are intersections of the hyperplane that corresponds to F'_4 and the curve in (y) that corresponds to C'_n . Hence order of the curve in (y) that corresponds to C'_n is $36d\beta/24$ or $\frac{1}{3}n$.

The order of the curve C'_{16mn} in (x) that corresponds to the intersection of H_m and H_n on Q may be reduced if either or both of H_m and H_n contain a plane of Q or the line VT . For example if H_m contains VT then the curve in (x) is $C_{16n(m-1)}$ and if H_m contains a plane of Q the curve in (x) is $C_{16n(m-2)}$.

Symmetric quartics. To a net of hyperplanes through a line s cutting Q in A and B corresponds in (x) a net of quartic surfaces with the same 4 double tangents $\phi_1=0$, $\phi_4=0$ and with two sets of 24 points each A' and B' corresponding to A and B as base points outside of the 8 F -points which are the points of tangency. When s is tangent to Q the quartic surfaces in (x) are all tangent to each other at 24 points. Now consider any two quadrics q' and q'' on Q . The common hyper-tangent planes of q' and q'' envelop two hyperquadric cones. Through a generic point of Q there are two tangent hyperplanes to each of the cones. To q' and q'' correspond in (x) two quartic surfaces F'_4 and F''_4 . Every tangent hyperplane of one of these cones cuts Q in a quadric which touches q' and q'' . To this quadric corresponds a quartic surface in (x) which touches F'_4 in 24 points, and F''_4 in 24 points. That is, given two symmetric quartic surfaces F'_4 and F''_4 there exist two systems of symmetric quartic surfaces such that every quartic of the system has 24 point contact with F'_4 and F''_4 .

To the intersection of a hyperplane through VT with Q corresponds in (x) a system of symmetric cubic surfaces $\phi_1^3 + \lambda_1\phi_1\phi_2 + \lambda_2\phi_3 = 0$. Let q' be a quadric not through VT . Now let I be the vertex of a hyperquadric cone through q' whose tangent hyperplanes cut Q in quadrics tangent to q' . To these correspond in (x) cubic surfaces and a quartic surface. Thus for a symmetric quartic surface corresponding to a generic quadric on Q there exists a system of cubic surfaces with the property of 24 point contact with the quartic surface.

SCALAR EXTENSIONS OF ALGEBRAS WITH EXPONENT EQUAL TO INDEX¹

SAM PERLIS

If a normal simple algebra A has a special structural property, it is of interest to inquire whether every finite scalar extension A_K also has this property. We shall make such an inquiry here where the property assumed for A is equality of exponent and index.

It suffices to consider only separable and purely inseparable fields K . Roughly stated, our result for separable extensions of finite degree is that preservation of the property in question depends only on whether it is preserved for scalar extension fields which are cyclic of prime degree. In the case of purely inseparable extensions the problem is immediately reducible to the case of such extensions of prime degree p , where p is the characteristic of the field. There remains the question whether such extensions always preserve equality of exponent and index, and we shall answer this question in the negative by means of an example.

Every p -algebra over a field of degree of imperfection² unity has equal index and exponent.³ The example mentioned above, however, shows that for every integer $r > 1$ there exists a modular field of degree of imperfection r such that not all the p -algebras ($p = 2$) over this field have exponent equal to index.

1. Exponent reduction factor. If A is any normal simple algebra of exponent ρ over F , the exponent of any scalar extension A_K is a divisor σ of $\rho = \sigma\tau$. The integer τ may be called the *exponent reduction factor* of A relative to K . This concept is analogous to that of index reduction factor and gives rise to a theorem analogous to that for index reduction factors.

THEOREM 1. *Let A be a normal simple algebra over F , and K be an algebraic extension of degree q over F . Then the exponent reduction factor of A relative to K is a divisor⁴ of q .*

PROOF. The direct power A^σ has exponent τ and K as splitting field. Now τ divides the index μ of A^σ , and μ divides the degree q of the splitting field K . Hence τ divides q .

¹ Presented to the Society, April 12, 1940.

² For the concept of degree of imperfection see §3 of O. Teichmüller, *p-Algebren*, Deutsche Mathematik, vol. 1 (1936), pp. 362–388.

³ Cf. O. Teichmüller, op. cit., p. 384. See also A. A. Albert, *p-algebras over a field generated by one indeterminate*, this Bulletin, vol. 43 (1937), pp. 733–736.

COROLLARY. *If the degree q of K is prime to the exponent of A , the scalar extension A_K has the same exponent and index as A .*

For purely inseparable fields K the following result is stronger than that of Theorem 1.

THEOREM 2. *The exponent reduction factor of A relative to a purely inseparable field K is a divisor of the exponent of K .*

PROOF. Now F has characteristic p , K has exponent p^a and degree p^b over F , and $A = B \times B_1$ where B has exponent p^a and B_1 has exponent prime to p . We have $\rho = p^a m = \sigma \tau$, where σ is the exponent of A_K . By Theorem 1, τ is a power of p so that $\sigma = p^f m$, B_K has exponent p^f , and $\tau = p^{e-f}$. Hence B_K has a purely inseparable splitting field L of exponent p^f over K . Now L splits B and is purely inseparable of exponent at most p^{f+a} over F , whence it follows that $p^{f+a} \geq p^a$, and $p^a \geq p^{e-f} = \tau$, as desired.

2. **Separable extensions.** In view of the fundamental result on the factorization of a normal division algebra corresponding to the factorization of its degree, it is completely sufficient to consider algebras of prime-power index p^a , and we shall do so. We begin with a known tool theorem⁴ on fields.

LEMMA 1. *Let K be a separable field of degree $p^a g$ over a field W , with g prime to p . Then K is contained in a field L of degree $p^a h$ over W , with h prime to p , such that*

$$(1) \quad L = L_e > L_{e-1} > \cdots > L_0 \geq W,$$

where L_i is cyclic of degree p over L_{i-1} ($i = 1, 2, \dots, e$) and L_0 is separable of degree h over W .

This result will now be applied to algebras.

THEOREM 3. *Let F be a field, p a prime, and W any separable finite extension of F . Every normal simple algebra A of index p^n and exponent p^n (n variable) over every such field W has the property that the index and exponent of A_K are equal, for every separable field K of finite degree over W , if and only if this is true for every field K which is cyclic of degree p over W .*

⁴ Cf. the proof of Theorem 31, Chapter IV of A. A. Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939. A brief proof is obtainable as follows. The field K is contained in a normal field N of degree $p^a h$ over W , h prime to p . Then N is normal over K and contains a Sylow subfield L of degree hg^{-1} over K , degree $p^a h$ over W . Hence N is metacyclic over a subfield L_0 , and we have (1).

PROOF. We simply apply Lemma 1 and Theorem 1. Since the degree of L over K is prime to p , the index and exponent of A_K are the same, respectively, as those of A_L . Now $A_L = (A_{L_0})L$ and A_{L_0} has index and exponent both equal to p^n since L_0 has degree h over W . By assumption A_{L_1} has equal index and exponent. But L_1 is a separable finite extension of F and L_2 is cyclic of degree p over L_1 so that A_{L_2} has equal index and exponent. A finite number of steps of this sort completes the proof.

Every finite extension of a field W has the form

$$(2) \quad Q = Q_t > Q_{t-1} > \cdots > Q_0 = K \geq W,$$

where each Q_i is purely inseparable of prime degree q over Q_{i-1} , q the characteristic of F , and Q_0 is separable over W . If $q \neq p$, the index and exponent of A_Q are the same as those of A_K , the type of scalar extension considered in Theorem 3. If $q = p$, we obtain an analogue of Theorem 3, applying to all finite extension fields Q , by allowing W to vary over all finite extensions of F , and assuming that A_Q has equal index and exponent not only for all fields Q which are cyclic of degree p over W but also for all fields Q which are purely inseparable of degree p . It follows that every A_Q has index equal to exponent, Q any finite extension of F .

3. Purely inseparable extensions. Let P be either the prime field of characteristic two or a field obtained from this prime field by adjunction of a finite number of independent indeterminates. Let x_0 and x be independent indeterminates over P , and $F = P(x_0, x)$. We shall construct an example of a cyclic algebra

$$(3) \quad D = (Z, S, x_0x)$$

over F with the properties described in Theorem 4 below.

To select the cyclic extension Z we first consider $F_0 = P(x_0)$. The equation $\lambda^2 = \lambda + x_0$ is irreducible over F_0 , hence defines⁵ a cyclic field $Y_0 = F_0(\xi)$ of degree two over F_0 , and Y_0 is contained in a field Z_0 which is cyclic of degree four over F_0 . Now $Z = Z_0(x)$ is cyclic of degree four over $F = F_0(x)$ and Z contains $Y = Y_0(x)$. Any generating automorphism S of Z_0 over F_0 may be regarded as a generating automorphism of Z over F .

If the exponent of $D = (Z, S, x_0x)$ were less than four, we should⁶ have $D^2 \sim (Y, S, x_0x) \sim 1$ so that x_0x would be a norm in Y over F ,

⁵ For the theory of cyclic fields used in this section see Chapter IX of A. A. Albert, *Modern Higher Algebra*, Chicago, 1937.

⁶ Albert, *Structure of Algebras*, chap. 7, Theorem 14.

contrary to the fact that x is an indeterminate over F_0 . Hence D has exponent and index four and D is a division algebra.

The field $K = F(j)$, $j^2 = x$, is purely inseparable of degree two over F and

$$(4) \quad D_K = (Z_K, S, x_0x).$$

Now $Y = F(\xi)$, $\xi^2 = \xi + x_0$ and, letting $Y_1 = Y \times K$, we have $x_0 = N_{Y|F}(\xi) = N_{Y_1|K}(\xi)$, $x_0x = N_{Y_1|K}(\xi j)$. The exponent of D_K is thus at most two while by Theorem 1 it is at least two; hence it is equal to two. However, when the extension Z_0 is appropriately chosen we may show by a matrix representation of D that D_K has index four. To make the required proof we need only show that K is not equivalent over F to a subfield of D .

In order to select Z_0 we prove

LEMMA 2. *The extension Z_0 over Y_0 may be chosen so that x_0 is not a norm with respect to Z_0 over Y_0 .*

We have

$$(5) \quad Z_0 = Y_0(\eta), \quad \eta^2 = \eta + x_0\xi + w,$$

where w in F_0 is at our choice. We shall choose $w=1$. Suppose $x_0 = N_{Z_0|Y_0}(z)$, z in Z_0 . Then z cannot be in Y_0 for, if so, $z^2 = x_0$ whereas Y_0 contains no quantities inseparable over F_0 . Hence $z = a_1(b + \eta)$ with $a_1 \neq 0$, a_1 and b in Y_0 , $x_0 = N_{Z_0|Y_0}(z) = a_1^2(b^2 + b + x_0\xi + 1)$ and, if $a = a_1^{-1}$,

$$(6) \quad b^2 + b + x_0\xi + 1 = x_0a^2.$$

Letting $a = \alpha_1\xi + \alpha_2$, $b = \beta_1\xi + \beta_2$ with α_i and β_i in F_0 we find that (6) is equivalent to the pair of equations

$$(7) \quad \begin{aligned} \beta_1^2 + \beta_1 + x_0 &= \alpha_1^2 x_0, \\ \beta_2^2 + \beta_2 + 1 &= x_0(\alpha_2^2 + \beta_1 + x_0). \end{aligned}$$

Let

$$(8) \quad \alpha_i = \mu_i/\delta_i, \quad \beta_i = \nu_i/\delta'_i, \quad i = 1, 2,$$

be expressions for the α_i and β_i as quotients of relatively prime polynomials in $P[x_0]$ with denominators δ_i and δ'_i all monic (i.e., having leading coefficients unity). Substituting in (7) we find that $\delta'_1 = \delta_1$ so that (7) is equivalent to

$$\nu_1^2 + \nu_1\delta_1 + x_0\delta_1^2 = x_0\mu_1^2.$$

This equation shows that δ_1 is prime to x_0 .

Now substituting from (8) in (7₂) we are led to the equation

$$(9) \quad \nu_2^2 + \nu_2 \delta'_2 + \delta_2'^2 = x_0 \delta_2'^2 (\mu_2^2 \delta_1 + \nu_1 \delta_2^2 + x_0 \delta_1 \delta_2^2) / \delta_2^2 \delta_1.$$

We observe from (9) that δ_2^2 divides $x_0 \delta_1 \delta_2'^2$, and since δ_1 is prime to x_0 , it follows that if δ_2 has a factor x_0 so does δ_2' . Thus in any case the right side of (9) is a polynomial divisible by x_0 . If the constant terms of ν_2 and δ_2' are ν and δ , respectively, we see then that the left side of (9) has constant term $\nu^2 + \nu\delta + \delta^2 = 0$. Further, ν and δ cannot both be zero since ν_2 and δ_2' are relatively prime. The equation $\lambda^2 + \lambda + 1 = 0$ thus has a solution $\lambda = \nu\delta^{-1}$ or $\lambda = \delta\nu^{-1}$ in P . On the other hand this equation has no solution in the field P , and this contradiction establishes the lemma.

THEOREM 4. *There exists a cyclic algebra D of index and exponent four over an appropriate field F of characteristic two, and a purely inseparable extension K of degree two over F , such that D_K has index four and exponent two.*

To make the proof we first represent $D = (Z, S, x_0x)$ by a set of four-rowed matrices with elements in Z . We have $D = Z + uZ + u^2Z + u^3Z$, $u^4 = x_0x$, and $zu = uz^S$ for every z of Z . If

$$(10) \quad v = z_0 + uz_1 + u^2z_2 + u^3z_3, \quad z_i \text{ in } Z,$$

is any quantity of D and U is the vector $(1, u, u^2, u^3)$, then $vU = U\bar{v}$ where $\bar{v}$ is the matrix

$$(11) \quad \bar{v} = \begin{pmatrix} z_0 & x_0 z_3^S & x_0 z_2^S & x_0 z_1^S \\ z_1 & z_0^S & x_0 z_3^S & x_0 z_2^S \\ z_2 & z_1^S & z_0^S & x_0 z_3^S \\ z_3 & z_2^S & z_1^S & z_0^S \end{pmatrix}.$$

The correspondence $v \rightarrow \bar{v}$ is an equivalence over F (when we identify F with the field of four-rowed scalar matrices over F) of D with the set of matrices $\bar{v}$. Note that the quantities $v = z_0$ in z correspond to diagonal matrices, $\text{diag} \{z_0, z_0^S, z_0^{S^2}, z_0^{S^3}\}$, such that $|\bar{v}| = N_{Z|F}(z_0)$. For this norm we shall use the simpler symbol $N(z_0)$.

Suppose that D contains a subfield equivalent over F to K , that is, D contains a quantity v such that $v^2 = x$. Then $\bar{v}^2 = x$, $|\bar{v}^2| = x^4$, and $|\bar{v}| = x^2$. Every z_i in (10) has the form $z_i = s_i t_i^{-1}$ where s_i and t_i are in the polynomial domain $Z_0[x]$ and are relatively prime. Also, $N(s_i)$ and $N(t_i)$ are in $F_0[x]$.

Let x^m be the highest power of x occurring as a factor in the denominators t_i . Then $w = vx^m = z'_0 + uz'_1 + u^2z'_2 + u^3z'_3$, $z'_i = z_i x^m = s'_i (t'_i)^{-1}$, t'_i prime to x ($i=0, 1, 2, 3$), and w corresponds to a matrix $\bar{w} = \bar{v}x^m$ all of whose elements have denominators prime to x . Further, $|\bar{w}| = |\bar{v}|x^{4m} = x^{2+4m}$, and $N(t'_i)$ is in $F_0[x]$ and is prime to x .

We shall prove $m=0$. Assuming $m>0$ we have

$$(12) \quad |\bar{w}| = x^{2+4m} = N(z'_0) + xx_0Q = \frac{N(s'_0)}{N(t'_0)} + xx_0Q,$$

where Q may be written as a quotient of polynomials in x with denominator prime to x . From the equation,

$$(13) \quad N(t'_0)x^{2+4m} = N(s'_0) + xx_0Q \cdot N(t'_0),$$

we see that x is a factor of $N(s'_0)$, hence of s'_0 . This result implies that x^m is not a factor of t'_0 . Using $s'_0 = xs'_0''$ in the matrix $\bar{w}$ we find

$$(14) \quad |\bar{w}| = x^{2+4m} = \frac{N(s'_1)}{N(t'_1)} xx_0 + x^2Q_1,$$

and as above we deduce that $s'_1 = xs'_1''$ with s'_1'' in $F_0[x]$.

We may continue this process until we have shown that every s'_i is divisible by x . But this means that no t_i has the factor x^m , contrary to the definition of m . This contradiction proves that $m=0$ so that the denominators t_i of all the z_i are prime to x . Now $w=v$, and by repeating the argument above we find that s_0 and s_1 are divisible by x .

Computing $v^2 = a_0 + ua_1 + u^2a_2 + u^3a_3 = x$ (a_i in Z), we must obtain $a_1 = a_2 = a_3 = 0$, $a_0 = x$. For a_0 we find

$$(15) \quad a_0 = z_0^2 + x_0x(z_1^S z_2 + z_2^S z_1 + z_3^S z_1) = x,$$

whence we have

$$(16) \quad z_0^2 + x_0x(z_1^S z_2 + z_1^S z_3) = x(1 - x_0z_2z_2^S).$$

Since the numerators of the terms on the left are all divisible by x^2 , the numerator, $t_2t_2^S - x_0s_2s_2^S = N_{Z|F}(t_2) - x_0N_{Z|F}(s_2)$, of the quantity in parentheses on the right must have a factor x . If the polynomials t_2 and s_2 of $Z_0[x]$ have constant terms τ_2 and σ_2 , respectively, the constant term of $N_{Z|F}(t_2) - x_0N_{Z|F}(s_2)$ is, then,

$$(17) \quad N_{Z|F}(\tau_2) - x_0N_{Z|F}(\sigma_2) = 0.$$

Since t_2 is prime to x , we have $\tau_2 \neq 0$, hence $\sigma_2 \neq 0$ by (17). Also, τ_2 and σ_2 are in Z_0 . Thus (17) may be written as

$$(18) \quad \bar{x}_0 = N_{\mathbb{Z}_0|\mathbb{F}_0}(\tau_2 \sigma_2^{-1}),$$

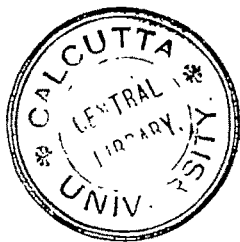
contrary to the result stated in Lemma 2. This contradiction completes the proof.

Let r be any integer greater than unity, and let P be obtained from its prime subfield P_0 by adjunction of $r-2$ independent indeterminates. Then F is obtained from the perfect field P_0 by adjunction of r indeterminates, F and K are said to be fields of degree of imperfection r , and we have the following result.

THEOREM 5. *For every integer $r > 1$ there exists a modular field K of degree of imperfection r such that not all the p -algebras over K have exponent equal to index.*

This is in contrast to the case $r=1$ for which every p -algebra is known to be cyclic with equal index and exponent. One may note, finally, that much of the work above is valid when the characteristic is any prime p , and it seems likely that the remaining details can be carried through for this generalized case.

ILLINOIS INSTITUTE OF TECHNOLOGY





BOOK REVIEWS

Elements of Statistical Reasoning. By Alan E. Treloar. New York, Wiley; London, Chapman & Hall, 1939. 11+261 pp. \$3.25.

The purpose of this book seems to be to develop fundamental statistical concepts for those who wish to reason carefully rather than to provide a compendium of statistical techniques. The author adopts the rather sensible viewpoint that it is from generous rather than meager sets of data that one grasps the general principles underlying the analysis of variation systems, and as a consequence his "original plan to include a critical analysis of small-sample techniques was finally laid aside." From this it should not be inferred that the book contains obsolete methods. On the contrary it treats such topics as chi squared and the sampling distribution of the correlation coefficient. In connection with the latter it even gives a brief discussion of the use of the t distribution for small samples from a population in which the correlation is zero.

Some other topics discussed are binomial, normal, and Poisson series, proportions (including proportions of vital statistics) and their sampling errors, contingency tables. Particular mention might be made of the discussion of the two types of error arising in making tests of significance: (A) insignificance is claimed when significance exists, (B) significance is claimed when it does not exist.

The material is clearly presented, and those who wish to obtain an introduction to modern statistical inference can study the book with profit.

PAUL R. RIDER

Tables of Sine, Cosine and Exponential Integrals. Prepared by the Federal Works Agency Work Projects Administration for the City of New York, A. N. Lowan, Technical Director. National Bureau of Standards, Washington, D. C. Vol. 1, 1940. 24+444 pp. \$2.00. Vol. 2, 1940. 37+225 pp. \$2.00.

This is the fifth and sixth volumes in the series of mathematical tables prepared by the Project for the Computation of Mathematical Tables under the auspices mentioned in the title.

The tabulations of the functions $Si(x)$, $Chi(x)$, $Ei(x)$, and $-Ei(-x)$ over the range from 0 to 2 at intervals 0.0001 in Volume 1 and from 0 to 10 at intervals of 0.001 in Volume 2 are given to nine places of decimals. These tables go a long way toward placing these functions in the class of known functions. Ordinarily, simple linear interpolation

is adequate. Even for nine place accuracy, at most second differences (as tabulated) are required either for direct or inverse interpolation by the effective methods explained and illustrated in the introduction to the two volumes. The functions tabulated are

$$\begin{aligned} Si(x) &= \int_0^x \frac{\sin t}{t} dt; & Ci(x) &= \int_{\infty}^x \frac{\cos t}{t} dt; \\ Ei(x) &= \int_{-\infty}^x \frac{e^t}{t} dt; & -Ei(-x) &= \int_x^{\infty} \frac{e^{-t}}{t} dt. \end{aligned}$$

These tables supplement those recently given out by the British Association for the Advancement of Science by using a much smaller interval of the argument. Each table is preceded by a detailed explanation of the method of calculation and followed by a bibliography of tables and one of applications.

At the end of the second volume are supplementary tables of

$$Si(n\pi + h), \quad n = 1, 2, 3; \quad Ci\left(\frac{k\pi}{2} + h\right), \quad k = 1, 3, 5.$$

The Tables are reproduced by the photo-offset process from type-written sheets. There are fifty entries on each page, arranged in groups of five, followed by a leaded interval. The result is a clear, easily read page. The actual work was done under the supervision of six subdirectors by a staff of some 250 computers. Such use has been made of the checks and controls described in the introduction that a very high degree of accuracy has been attained.

VIRGIL SNYDER

NOTES

At the meeting of the American Association for the Advancement of Science at the University of New Hampshire at Durham on June 26-27, 1941, the following addresses were presented: *Statistical mechanics*, by Professor Norbert Wiener of Massachusetts Institute of Technology; *Topological transformation groups in euclidean spaces*, by Professor Deane Montgomery of Smith College; *Vector lattices*, by Professor Garrett Birkhoff of Harvard University; *Some recent researches on the coloring of maps*, by Professor D. C. Lewis of the University of New Hampshire.

At New York University five tuition-free courses in higher mathematics, approved by the United States Office of Education, were given in the evenings beginning June 12. The subjects included advanced methods of applied mathematics, fluid dynamics and applications, theory of elastic plates, electrical network theory and vibrations.

Dr. Harold Davenport of the University of Manchester has been awarded The Adams Prize of the University of Cambridge.

Professor L. J. Mordell of the University of Manchester has been awarded the De Morgan Medal of the London Mathematical Society.

At the fiftieth anniversary celebration of the University of Chicago in September Professor G. D. Birkhoff of Harvard University and Professor Oswald Veblen of the Institute for Advanced Study were awarded honorary doctorates.

The degree of doctor of science has been conferred upon Professor P. W. Bridgman of Harvard University by Stevens Institute of Technology.

An honorary doctorate of science has been conferred by Harvard University and by the Polytechnic Institute of Brooklyn upon Dr. Vannevar Bush, president of the Carnegie Institution of Washington.

An honorary doctorate of science has been conferred by Princeton University upon Professor Emeritus L. E. Dickson of the University of Chicago.

Dean H. E. Hawkes of Columbia University received an honorary doctorate of laws from Bethany College.

Professor J. W. Lasley of the University of North Carolina has been elected president of the Southeastern Section of the Mathematical Association of America.

Dr. Alfred Tarski of Harvard University has been awarded a Guggenheim Fellowship.

Dr. E. G. Begle and Dr. E. R. Kolchin have been awarded National Research Fellowships. They will study at the University of Michigan and the Institute for Advanced Study, respectively.

The William Lowell Putnam Scholarship for 1941 for study at Harvard University has been awarded to Mr. R. F. Arens of the University of California at Los Angeles.

Dr. J. H. Bell of the University of Wisconsin has been appointed research assistant by the National Research Council of Canada.

Assistant Professor P. O. Bell of the University of Kansas has been appointed to a lectureship at Princeton University.

Assistant Professor A. C. Berry of Columbia University has been appointed to an associate professorship at Lawrence College, Appleton, Wisconsin.

Assistant Professor Augusto Bobonis of the University of Puerto Rico has been promoted to an associate professorship.

Dr. C. E. Clark of Purdue University has been promoted to an assistant professorship.

Mr. Paul Eberhart of Washburn Municipal University of Topeka, Topeka, Kansas, has been promoted to an assistant professorship.

Dr. W. K. Feller of Brown University has been promoted to an assistant professorship.

Dr. J. W. Green of the University of Rochester has been promoted to an assistant professorship.

Dr. E. H. Hadlock of Cornell University has been appointed to an assistant professorship at Hastings College, Hastings, Nebraska.

Dr. Marshall Hall of Yale University has been promoted to an assistant professorship.

Dr. H. J. Hamilton of Pomona College has been promoted to an assistant professorship.

Dr. I. E. Highberg of Whitman College, Walla Walla, Washington, has been promoted to an assistant professorship.

Dr. Mabel G. Humphreys of Newcomb College has been promoted to an assistant professorship.

Assistant Professor E. D. Jenkins of Eastern Kentucky State Teachers College has been promoted to an associate professorship.

Assistant Professor S. C. Kleene of the University of Wisconsin has been appointed to an associate professorship at Amherst College.

Mr. R. A. McKean of Columbia University has been appointed to a lectureship at Polytechnic Institute of Brooklyn.

Associate Professor H. B. MacDougal of South Dakota State College has been promoted to a professorship and will be acting head of the department of mathematics.

Professor Szolem Mandelbrojt of the College of France will be visiting professor at Rice Institute during this academic year.

Assistant Professor Anna E. Many of Newcomb College has been promoted to a professorship.

Associate Professor R. H. Marquis of Ohio University has been promoted to a professorship.

Dr. R. J. Michel of the University of Missouri has been appointed to a professorship at Southeast Missouri State Teachers College.

Miss Florence L. Munroe of Northampton High School, Northampton, Massachusetts, has retired.

Dr. E. S. Quade of the University of Florida has been promoted to an assistant professorship.

Associate Professor Rafael Sánchez-Díaz of the College of Agriculture and Mechanics Arts, University of Puerto Rico, has been promoted to a professorship.

Dr. Arthur Sard of Queens College has been promoted to an assistant professorship.

Dr. A. C. Schaeffer of Stanford University has been promoted to an assistant professorship.

Assistant Professor I. J. Schoenberg of Colby College has been appointed to an assistant professorship at the University of Pennsylvania.

Mr. C. E. Stevens of Hofstra College, Hempstead, New York, has been promoted to an assistant professorship.

Dr. M. S. Webster of Purdue University has been promoted to an assistant professorship.

Associate Professor C. H. Wheeler of the University of Richmond has been promoted to a professorship and will be head of the department of mathematics.

The following appointments to instructorships are announced: Case School of Applied Science: Dr. E. L. Crow, Mr. P. E. Guenther; University of Florida: Mr. R. D. Specht; Harvard University: Dr. Irving Kaplansky, Dr. C. E. Rickart; University of Maryland: Mr. E. N. Nilson; University of Michigan: Dr. C. J. Everett; Northwestern University: Mr. E. L. Buell, Mr. L. L. Cronvich, Mr. R. H. Stark; Ohio State University: Dr. E. J. Mickle; Pennsylvania State College: Dr. W. J. Harrington; Princeton University: Dr. Warren Ambrose, Dr. H. A. Arnold; University of Richmond: Mr. E. S. Grable; Smith College: Dr. Jeanne S. Le Caine; Stanford University: Mr. G. E. Forsythe; United States Naval Academy: Dr. Byron Cosby, Mr. C. B. Lindquist, Dr. A. W. McGaughey, Dr. Seymour Sherman; Vassar College: Alexandra I. Forsythe; Wellesley College: Miss Katharine E. Hazard.

Professor G. Prange of the Technical School in Hannover died February 3, 1941.

Professor D. D. Leib of Connecticut College died June 15, 1941. He had been a member of the Society since 1909.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

330. A. A. Albert: *Division algebras over a function field.*

Let D be a division algebra of degree m over its centrum K which is algebraic of finite degree over $L(x)$. The author studies the least degree n of fields W of finite degree over L such that a composite of K and W splits D . Clearly $n \geq m$ and one asks whether in general $n = m$. The answer is in the negative, since if L is finite and $K = L(x)$, then n may be any integer divisible by m and such that every prime factor of n divides m . The property $n > m$ is not a function of the finiteness of L and this is indicated by proving it for $m = 2$ and all fields L such that there exist at least two inequivalent quadratic extensions of L . In particular when L has characteristic two this result implies $n > m$ also for any $m = 2^r$. The paper mentions finally an unpublished master's dissertation of Louis Gordon showing that if $m = 4$ and K is algebraic of finite degree over $L(x_1, \dots, x_r)$ then D may be non-cyclic if $r > 2$ and the x_i are independent indeterminates over L . However, it is proved that D is necessarily cyclic for $r = 2$. (Received July 14, 1941.)

331. A. A. Albert: *Non-associative algebras. I.*

The paper presents a new general theory of non-associative algebras. Any algebra A is a linear space L of finite order over a field. Multiplication in A is then defined by a linear mapping $x \rightarrow R_x$ of L on the space of right multiplications R_x of A . A second algebra A_0 of the same order may be regarded as consisting of the same quantities as does A but with multiplication defined by $x \rightarrow R_x^{(0)}$. Then A and A_0 are called isotopes, or isotopic algebras, if there exist non-singular linear transformations S, T, U on L such that $R_x^{(0)} = SR_xT, y = x^T$. Isotopy is then a widening of the concept of equivalence except in the case of associative algebras with a unity quantity where the concepts coincide. This possible widening accounts at least partly for the inordinately large number of algebras of low orders. It is shown that there exist non-commutative isotopes with a unity quantity of commutative non-associative algebras with a unity quantity, and non-simple isotopes of simple algebras. Related topics inspired by isotopy and connections of the theories of associative algebras, Lie algebras and Jordan algebras are studied. (Received July 14, 1941.)

332. Reinhold Baer: *Automorphism rings of primary abelian operator groups.*

The representation of a ring as the ring of all the automorphisms of a primary abelian operator group expresses significant inner properties of this ring, since there

exists essentially at most one such representation of a given ring. Thus it is the object of this investigation to expose invariant qualities of these automorphism rings. They are found to be peculiarities of their ideal-theory; and amongst these are listed a number which form a complete set of postulates for the class of all the automorphism rings of primary abelian operator groups, thus proving incidentally that these rings are completely determined by their ideal-theory. The properties of the ideals in these rings reflect so completely the structure of the underlying operator group that it is possible to prove the essential identity of the group of automorphisms of the ring and of the group of projectivities of the system of admissible subgroups of the underlying operator groups. (Received July 31, 1941.)

333. Richard Brauer: *Investigations on group characters.*

The paper continues the previous work of C. Nesbitt and the author on modular characters of groups and their connection with the ordinary group characters. Let G be a group of finite order $g = p^a g'$ where p is a prime which does not divide g' . Let F be an ordinary irreducible representation of G whose degree is divisible by p^{a-1} . It is shown that the representations belonging to the block B of $F \pmod{p}$ have the same property. The decomposition $\pmod{p}$ of the characters of B can be described by a linear graph which is proved to be a tree. As an application, it can for instance be shown that a simple group of an order $4p^a q^b$ (p, q primes, $a \leq 2$) must have order 60; a simple group of an order $3p^a q^b$ (p, q primes, $a \leq 2$) must have the order 60 or 168. (Received July 29, 1941.)

334. Richard Brauer: *On the indecomposable representations of algebras.*

The indecomposable representations M of an algebra A are studied, that is, the representations M for which the underlying vector space is not the direct sum of two invariant subspaces. The necessary and sufficient conditions are obtained that M is an end constituent of the regular representation of A . In addition, cases are studied in which A has infinitely many non-equivalent indecomposable representations. (Received July 29, 1941.)

335. R. H. Bruck: *The number of absolute invariants of a tensor.*

From the method of L. P. Eisenhart (*Continuous Groups of Transformations*, pp. 62 and 20) it is deduced for the complex field that the number of functionally independent absolute invariants, under $GL(n)$, of a tensor $T_{i_1 i_2 \dots i_r}^{(a)} = T_{i_1 i_2 \dots i_r}^{a_1 a_2 \dots a_r}$ is equal to $N - Q$, where N is the number of components of $T_{i_1 i_2 \dots i_r}^{(a)}$ and Q is a two-way rank of the tensor $\xi_{\alpha\beta}^{(a)} = [\delta_{\alpha_1}^{\beta_1} T_{i_1 i_2 \dots i_r}^{a_1 a_2 \dots a_r} + \dots + \delta_{\alpha_r}^{\beta_r} T_{i_1 i_2 \dots i_r}^{a_1 a_2 \dots a_r}] - [\delta_{\alpha_1}^{\beta_1} T_{i_1 i_2 \dots i_r}^{a_1 a_2 \dots a_r} + \dots + \delta_{\alpha_r}^{\beta_r} T_{i_1 i_2 \dots i_r}^{a_1 a_2 \dots a_r}]$. The problem of finding Q reduces to the purely algebraic one of determining the number $n^2 - Q$ of linearly independent tensors F_{β}^{α} satisfying (*) $\xi_{\alpha\beta}^{(a)} F_{\beta}^{\alpha} = 0$. The equations (*) may be replaced by the sets of simultaneous equations obtained by substituting in turn for $T_{i_1 i_2 \dots i_r}^{(a)}$ each of its symmetrized parts (T. L. Wade, *American Journal of Mathematics*, vol. 63 (1941), pp. 645-657). The method is applied to various tensors. (Received July 30, 1941.)

336. R. H. Bruck and T. L. Wade: *On a tensor algebra of Cramlet.*

In this paper some concepts and methods of the general theory of linear associative algebras of tensors developed recently by the authors (*Bisymmetric tensor algebra*, Parts I and II, to appear in *American Journal of Mathematics*; see also this Bulletin,

abstracts 47-5-184 and 47-5-185) are extended. This extended form of general tensor algebra is related to a particular tensor algebra considered by C. M. Cramlet (*Annals of Mathematics*, (2), vol. 31 (1930), pp. 134-150) in connection with differential forms. (Received July 26, 1941.)

337. H. H. Campaigne: *An example of a cogroup.*

An example is given which answers a question of J. E. Eaton (*Theory of cogroups*, *Duke Mathematical Journal*, vol. 6 (1940), pp. 101-197). It is shown that this cogroup cannot be derived from any group by any conjugation. (Received July 25, 1941.)

338. A. D. Campbell: *On the application of barycentric coordinates to lattice theory.*

One can label the elements of any specific lattice with barycentric coordinates in such a way that the least upper bound and the greatest lower bound of any two elements can be read off from the subscripts of these barycentric coordinates. Also by using abstract sets for these subscripts (instead of integers) many old and new results in lattice theory can be proved quickly. If P_α is one element of a lattice and P_β another element then $P_\alpha \cap P_\beta = P_{\alpha \cdot \beta}$, where $\alpha \cdot \beta$ is the meet of the sets α and β . Also $P_\alpha \cup P_\beta = P_{\alpha, \beta, a_{\alpha, \beta}}$ where $\alpha, \beta, a_{\alpha, \beta}$ means the join of the sets α, β and $a_{\alpha, \beta}$ and where $a_{\alpha, \beta}$ is a set whose presence in the subscript is caused by the presence of the join α, β . The necessary and sufficient condition for a lattice to be modular is: If $P_\alpha, P_\beta, P_\gamma$ are any three elements of this lattice such that $\alpha \leq \gamma$, then $\gamma \cdot (a_{\alpha, \beta} - a_{\alpha, \beta, \gamma}) = 0$. The necessary and sufficient conditions for a lattice to be distributive reduce to $\alpha \cdot (a_{\beta, \gamma} - a_{\alpha, \beta, a_{\alpha, \gamma}}) = 0$, where $P_\alpha, P_\beta, P_\gamma$ are any three elements of the lattice. These barycentric coordinates seem to offer great possibilities in the handling of lattice theory. (Received July 2, 1941.)

339. P. W. Carruth: *Arithmetic of ordinals with applications to the theory of ordered abelian groups.*

Hessenberg has defined a "natural sum" of ordinals, and Hausdorff has defined a "natural product" of ordinals. This paper gives new characterizations of these operations. It is shown that Hessenberg's natural sum of two ordinals α and β is the best bound for the order type of the join of two well-ordered subsets, of respective types α and β , of an ordered set. Hausdorff's natural product of two ordinals α and β is shown to be the best bound for the order type of a certain rectangular array of ordered elements that has " α rows" and " β columns." Another application of Hausdorff's natural product is given in determining a bound for the order type of the semi-group generated by a well-ordered subset of positive elements of an ordered abelian group. (Received June 6, 1941.)

340. R. J. Duffin and R. S. Pate: *Generalized groups and the Jordan-Hölder theorem.*

Generalized groups (hypergroups, quasigroups, groupoids, and so on) arise by removing some of the group postulates. For a commutative group the Jordan-Hölder theorem concerns all subgroups, but if the commutative postulate is removed the theorem concerns only subgroups having a commutative behavior. Likewise the generalized groups have a Jordan-Hölder theorem provided every postulate removed from the group as a whole is transferred in a suitable manner to the "subgroups" con-

cerned. Most generally the "group" becomes nothing more or less than a non-commutative, non-associative, multiple valued product. It is convenient to restore the unique product by adjoining subsets of elements to the original elements. The generalized group becomes then a partially ordered set each of whose elements effects by the product operation an order preserving automorphism. It is to be emphasized that the Jordan-Hölder theorem for such a system is more than an analogy with the theorem for groups; indeed the theorem for groups is a direct consequence of the generalized theorem. For a commutative group, what is meant by the Jordan-Hölder theorem is unambiguous, but for a non-commutative group the theorem applies to both principal series and to composition series. (Received July 30, 1941.)

341. Fritz Herzog: *On an application of a theorem by Burnside.*

The theorem, referred to in the title, states that if a Sylow subgroup of a finite group G is contained in the center of its normalizer, then G contains a normal subgroup the index of which is the order of that Sylow subgroup. This theorem is applied to obtain the result that a group of order g all of whose Sylow subgroups are abelian must be abelian itself if certain number theoretical relations between the prime factors of g are satisfied. As a conclusion from the latter theorem, the totality of all positive integers g is determined which have the property that all groups of order g are abelian. (Received July 30, 1941.)

342. S. A. Jennings: *Central chains of ideals in an associative ring.*

Let A, B be any two (not necessarily distinct) ideals of an associative ring R ; then the ideal $A \circ B$ is defined to be the ideal generated by all elements of the form $ab - ba$, where $a \in A, b \in B$. A chain of ideals of $R, R = M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} = 0$ terminating with the 0-ideal is a central chain if $M_\rho \circ R \subseteq M_{\rho+1}, \rho = 1, 2, \dots$. A ring with a central chain is said to have finite class. In particular every nilpotent ring is of finite class. For such a ring the lower central chain is defined by setting $R = H_1, H_\rho = H_{\rho-1} \circ R$. Properties of this chain analogous to those of the lower central series of a nilpotent group are obtained. Also the upper central chain is defined, corresponding to the upper central series in a group. A ring is said to be solvable if the chain defined by setting $R = R^{(0)}$ and $R^{(n)} = R^{(n-1)} \circ R^{(n-1)}$ terminates with the 0-ideal. Some criteria for a ring to have finite class and for solvability are obtained and some applications considered. (Received July 25, 1941.)

343. B. W. Jones: *On an extension of a theorem of Witt.*

Ernst Witt (Journal für Mathematik und Physik, vol. 176 (1937), pp. 31-48) proved that if, in any algebraic field F of characteristic $\neq 2$, there is a transformation $\mathfrak{T}$ in F taking the vector space $\mathfrak{S}_1 + \mathfrak{S}_2$ into $\mathfrak{S}_1 + \mathfrak{S}_2$, where $\mathfrak{S}_2$ and $\mathfrak{S}_2$ are vector spaces in F orthogonal to $\mathfrak{S}_1$, then there is such a transformation taking $\mathfrak{S}_2$ into $\mathfrak{S}_2$. This paper proves the corresponding result for a ring R_p of p -adic integers with p odd, where not only $\mathfrak{T}$ but its inverse is in R_p . The same result holds for any $\mathfrak{P}$ -adic ring over an algebraic field where $\mathfrak{P}$ is a prime ideal relatively prime to 2. These results may also be stated in terms of quadratic forms. (Received July 28, 1941.)

344. G. K. Kalisch: *On Jordan algebras.*

An algebra A over a field F will be termed a Jordan algebra in case there exists an associative algebra B over F with an involutorial anti-automorphism J such that A is isomorphic to the algebra of the J -symmetric elements of B . Let C be a distributive

algebra of finite order over a field F contained in the field R of all real numbers. Suppose that the scalar extension $C \times R$ is the Jordan algebra defined as above for B either (1) the algebra of all n -rowed real matrices, (2) the algebra of all n -rowed complex matrices, (3) the algebra of all n -rowed matrices with quaternionic elements. Then it is proved that in all three cases the given algebra C over F is a Jordan algebra, in cases (1) and (3) for a normal simple algebra S over F such that $S \times R = B$, and in case (2), for a simple algebra S over F with centrum $K = F(q)$ where $q^2 = a < 0$ in F , J is an involutorial anti-automorphism of the second kind, and $S \times R = B$. Finally a classification of such algebras is given by means of congruent matrices as in the Lie algebra theory of N. Jacobson. (Received July 21, 1941.)

345. Morris Marden: *The zeros of certain composite polynomials.*

In this article the relative location of the zeros of the polynomials $A(z) = \sum_{k=0}^n a_k z^k$, $B(z) = \sum_{k=0}^n b_k z^k$, $C(z) = \sum_{k=0}^n a_k B(k) z^k$ is discussed. It is proved, for example, that, if all the zeros of $A(z)$ lie in the ring $0 \leq r_1 \leq |z| \leq r_2 \leq \infty$ and if all the zeros of $B(z)$ lie in the connected region bounded by two circles $|z| = \lambda_1 |z - m|$ and $|z| = \lambda_2 |z - m|$, then all the zeros of $C(z)$ lie in the ring $R_1 \leq |z| \leq R_2$, where $R_1 = r_1 |B(0)/B(m)|$, $r_1 \lambda_1^*$, or r_1 and $R_2 = r_2$, $r_2 \lambda_2^*$, or $r_2 |B(0)/B(m)|$ according as $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, $0 \leq \lambda_1 \leq 1 \leq \lambda_2$, or $1 \leq \lambda_1 \leq \lambda_2$. A similar theorem is proved for the case that all the zeros of $A(z)$ lie in a half-plane. The theorems found here are generalizations of theorems due to Laguerre, Pólya, and Weisner. The results are also extended to entire functions of genre 0 or 1. (Received July 29, 1941.)

346. Deane Montgomery and Leo Zippin: *A theorem on Lie groups.*

This note gives and proves a partial generalization of the following theorem: Let G be a Lie group and let G^* be a compact subgroup of G . Then there exists an open set O including G^* which has the property that if H is a subgroup of G which is in O , then there exists an element g in G such that $g^{-1}Hg$ is in G^* . This can be regarded as a generalization of the theorem that a Lie group has no subgroup near the identity except the identity itself. (Received July 3, 1941.)

347. D. C. Murdoch: *Automorphisms and quasi-groups.* Preliminary report.

An abelian quasi-group is one which satisfies the law $(ab)(cd) = (ac)(bd)$. It is said to be right nilpotent if the series of successive right unit subquasi-groups terminates in an idempotent element. Every right nilpotent quasi-group is a direct product of cyclic quasi-groups. If it is both left and right nilpotent then there exists a composition series in which successive quotient groups are all associative, and therefore ordinary groups. These and other results may be translated into statements concerning automorphisms of abelian groups. (Received July 28, 1941.)

348. Ivan Niven: *The m th roots of a quaternion.*

Any quaternion which is not a real number has exactly m distinct m th roots. These are exhibited in terms of the quaternion and the complex roots of the real quadratic equation satisfied by the quaternion (involving the trace and norm as coefficients). A real number has infinitely many m th roots in quaternions if $m \geq 3$; if $m = 2$, there are infinitely many square roots or just two according as the real number is negative or not. The method depends upon the establishing of a real equation satisfied by the quaternion; such an equation must be divisible by the quadratic referred to above. (Received July 29, 1941.)

349. Rufus Oldenburger: *The factorization of polynomials.*

It is proved that a polynomial f in one variable factors for a field K into a product gh of relatively prime polynomials g and h if and only if a certain polynomial F associated with f can be written for K as a sum $G+H$, where G and H are related to g and h respectively in the same way as F is to f . The sum $G+H$ corresponds uniquely to the product gh . The reducibility of a polynomial in one variable, with respect to a field K , is thus transformed into the problem of splitting a second polynomial into a certain sum of polynomials. Using these results it is shown that a binary form B with coefficients in a field K can be written in a finite number of ways as a sum $U+M$ of binary forms with coefficients in K , where if K is extended to a closed field K' , the form U has a unique minimal representation with respect to K' , and M is a form of degree n , whose minimal number cannot be increased by the addition of a term aL^n , L linear, to M . The splitting of B into $U+M$ is invariant in the usual sense. (Received July 28, 1941.)

350. Gordon Pall: *Quaternions and sums of three squares.*

U. V. Linnik (Bulletin of the Academy of Sciences of the USSR, Mathematics Series, vol. 4 (1940), pp. 363–402) gives the following result: Let p be an odd prime, m large, $m \not\equiv 4k$ or $8k+7$, $(-m|p) = 1$, $x_0^2 \equiv -m \pmod{p}$. Then as x ranges over all proper pure quaternions $i_1x_1 + i_2x_2 + i_3x_3$ of norm m , every quaternion of norm p occurs as a left-divisor of $x_0 + x$. His proof has the following serious error, which is adjusted in this article: He supposes that the number of representations as a sum of three squares of a binary quadratic form of determinant D does not exceed $c_1 D^{\frac{1}{2}}$. (Received June 5, 1941.)

351. H. A. Simmons: *Classes of maximum numbers associated with two symmetric equations in n reciprocals.*

The terminology used in stating the results here is that which was employed in the author's article with W. E. Block (Duke Mathematical Journal, vol. 2, p. 317). The first equation considered is $\sum_{i=1}^n a_i (\pi(x))^{-i} = b/a$, ($a = (c+1)b-1$, $\pi(x) = x_1x_2 \cdots x_n$), in which b , c , and m are arbitrary positive integers, $n > 1$, and the a_i are any non-negative real numbers. The second equation is $\sum_{i=1}^n a_i (1/x) + \lambda \sum_{i=1}^{n-1} (1/x) + \mu \sum_{i=1}^n (1/x) = b/a$, in which a and b are arbitrary positive integers, $n > 2$, λ is a non-negative integer, and μ is a positive integer. The Kellogg-solution of each of these equations is shown to have the remarkable properties. These results are obtained through several innovations of the previous methods of identifying maximum numbers. A new lemma and a new term, "s-set," play important roles relative to each equation. Another element of newness in the argument relative to the second equation is the introduction for the sought maximum number of an upper bound analogous to an upper bound that Curtiss (American Mathematical Monthly, vol. 29, pp. 380–387) used in his solution of Kellogg's Diophantine problem. (Received July 28, 1941.)

352. Olga Taussky and Ernest Best: *A class of groups.*

A t -group G is a group in which every subgroup g_i self-conjugate in a subgroup g_j , itself self-conjugate in G , is self-conjugate in G . A t -group can also be defined as a group for which composition and chief series coincide. Abelian, hamiltonian and simple

groups are trivial examples. In this paper general properties of t -groups and a large class of non-trivial t -groups are investigated. This class includes those groups which have cyclic Sylow subgroups, in particular all groups with square-free order. (Received July 10, 1941.)

353. R. M. Thrall: *On symmetrized Kronecker powers and Lie representations of the full linear group.*

Let $\mathfrak{L}$ be the free Lie ring with n generators and denote by $\mathfrak{L}^i$ the ideal of $\mathfrak{L}$ generated by all monomials of degree i in the generators. The factor ring L^i/L^{i+1} serves as representation space for a representation $A \rightarrow L_i(A)$, called the i th Lie representation, of the full linear group, $GL(n)$. A recursion formula is obtained for trace $L_i(A)$ which expresses it as a sum of terms each the trace of a Kronecker direct product of symmetrized Kronecker powers of $L_j(A)$ with $j < i$. The first part of the paper is therefore devoted to a study of the analysis of the symmetrized Kronecker powers of an arbitrary representation of $GL(n)$. In the final sections of the paper the specific analysis of $L_i(A)$ is given for $i \leq 10$. The decomposition of the various Lie representations is closely associated with the determination of characteristic ideals of $\mathfrak{L}$. (Received July 23, 1941.)

354. T. L. Wade: *Euclidean concomitants of the ternary cubic.*

The results of C. M. Cramlet (this Bulletin, vol. 34 (1928), pp. 334-342) and of the writer (this Bulletin, vol. 47 (1941), pp. 475-478) which relate tensor algebra and invariant theory are used in the construction, reduction, and investigation of the geometric interpretations of euclidean concomitants of the ternary cubic $T_{abc}X^aX^bX^c$. Here $a, b, c = 1, 2, 3$ and T_{abc} is symmetric. (Received July 19, 1941.)

355. Louis Weisner: *Roots of certain classes of polynomials.*

From two polynomials $\phi(z)$ and $f(z) = \sum_{k=0}^n c_k z^k$, form the polynomial $g(z) = \sum_{k=0}^n c_k \phi(2k-n) z^k$. The author proves: (1) If the roots of $\phi(z)$ lie on the axis of pure imaginaries and the roots of $f(z)$ in the ring $r_1 \leq |z| \leq r_2$, then the roots of $g(z)$ lie in this ring. (2) If the roots of $\phi(z)$ lie in the half-plane $\operatorname{Re} z \leq 0$, and the roots of $f(z)$ in the circular region $|z| \leq r$, then the roots of $g(z)$ lie in this circular region. (Received June 20, 1941.)

356. L. R. Wilcox: *Complementation and modularity in lattices.*

Let L be a lattice with $0, 1$. If $a, b \in L$, then c is a left (right) complement within b of a in $a+b$ if $c \leq b$, $c+a = b+a$, and $(c, a) \perp ((a, c) \perp)$, where $(c, a) \perp$ means $ca = 0$ together with $(d+c)a = d+ca$ for every $d \leq a$. If for every $a, b \in L$ there exists a left (right) complement within b of a in $a+b$, then L is said to be left (right) complemented. Properties of these notions are studied and some of the results are the following. (1) If (a, b) is a modular pair, then c is a right complement within b of a in $a+b$ if and only if c is a right complement within b of ab in b . (2) If L is left complemented, then L is right complemented, the converse of (1) holds, and L is symmetric in the sense that the relation of modularity is symmetric. (3) If L is left complemented, and if S is the set of all products bc with (b, c) not modular, then $x \in S$, $y \leq x$ implies $y \in S$. A class of examples of left complemented lattices arising as subsets of a modular lattice is given, and the foundations of an extension theory for left complemented lattices are laid. (Received July 30, 1941.)



357. A. T. Brauer: *On the least quadratic non-residue.*

Let p be a prime and d the least quadratic non-residue (mod p). The best estimate for d was obtained by Vinogradov by analytical methods. This estimate, however, is valid only for sufficiently large primes. It is of interest to obtain such estimates also by elementary methods, in particular if these estimates are valid for all primes or for all primes of a fixed set. Gauss's first proof of the law of reciprocity gives such a bound for the primes of the form $8n+1$. His result was improved and generalized by Schur, Nagell, and the author. In this paper these results will be improved still further. Moreover the author obtains bounds for the least quadratic non-residue which is not divisible by d , and bounds for the number R of quadratic residues and for the number N of quadratic non-residues which are less than $p^{1/2}$. This latter result improves theorems of Pólya, Schur, and Landau for a special case. (Received July 29, 1941.)

358. Leonard Carlitz: *Note on generalized Bernoulli and Euler numbers.*

H. S. Vandiver (Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 555-559) has introduced certain numbers that he calls generalized Bernoulli numbers of the r th order. For the first order they are defined by $b_n(m, k) = (mb+k)^n$. For these numbers Vandiver proves a generalized Staudt-Clausen theorem. The present note contains a simple proof of this theorem using Lucas' method. By means of this method a theorem of the same type is proved for the generalized number $(m_1b+k_1)^{n_1} \cdots (m_rb+k_r)^{n_r}$ which is suggested by certain Bernoulli polynomials in several variables. For the corresponding Euler numbers congruences of Kummer's type are derived. (Received July 30, 1941.)

359. Leonard Carlitz: *The coefficients of the reciprocal of a series.*

A. Hurwitz (Mathematische Annalen, vol. 51 (1899), pp. 196-226 and *Mathematische Werke*, vol. 2, pp. 342-373) defined rational numbers E_m by means of $\wp(u) = 1/u^2 + \sum_{m=1}^{\infty} (2^{4m} E_m / 4m) (u^{4m-2} / (4m-2)!)$, where $\wp(u)$ is the Weierstrass function for the lemniscate case. Making use of the complex multiplication of $\wp(u)$ he proved a theorem of the Staudt-Clausen type for E_m . The present paper is concerned with series of the form $f(u) = \sum_{m=1}^{\infty} a_m u_m / m!$, where a_m is integral. Let the inverse of $f(u)$ be $\lambda(u) = \sum_{m=1}^{\infty} c_m u_m / m!$, and assume that c_m is a multiple of $(m-1)!$. Then theorems of the Staudt-Clausen type are proved for the coefficients in the expansion of $u/f(u)$ and $u^2/f^2(u)$. The latter includes Hurwitz' theorem (except for the partial fraction corresponding to the prime 2). The method is entirely elementary and makes use of certain general ideas on series of the form $f(u)$ due to Hurwitz. (Received July 30, 1941.)

360. J. M. Dobbie: *A certain method in additive number theory.* Preliminary report.

Ratio difference equations are used to generate identities involving sums of squares of integers. The method is that used by Carmichael in his lectures and follows a method of Cauchy (*Oeuvres*, series 1, vol. 8, pp. 42-115). It is related to the well known theta function method of Jacobi. Starting with the equation $u(x) = (1-tx)u(tx)$, $0 < |t| < 1$, the function $\prod_{n=1}^{\infty} [(1-t^n)^2(1-x)/(1+t^n)^2(1+x)] \prod_{n=1}^{\infty} (1-t^n x)(1-t^n x^{-1}) + (1+t^n x)(1+t^n x^{-1})$ is introduced and is expanded into a series about its poles. Special cases of the resulting identity are the classic identities for the forms a^2+b^2 , a^2+2b^2 , a^2+3b^2 . By raising $G(x, t)$ to powers, identities for treating the forms

$\sum_{i=1}^n a_i^2 + r \sum_{i=1}^n b_i^2$ (n any positive integer, $r=1, 2, 3$) are obtained. Other values of r can be treated by extending the function $G(x, t)$. This has been done for $r=4, 5, 7$, but the expansion for general r has not been obtained. The method also has been used to treat certain related forms. (Received July 29, 1941.)

361. R. D. James: *On the sieve method of Viggo Brun.*

A. A. Buchstab (Matematicheskii Sbornik, vol. 46 (1938), pp. 375-387, and Comptes Rendus de l'Académie des Sciences, URSS, vol. 29 (1940), pp. 544-548) has made improvements in the sieve method and so obtained more precise results. It is the purpose of this paper to point out that his method applies equally well to any infinite set of primes satisfying certain conditions. By applying the method to the set of primes congruent to 3 (mod 4), for example, it is shown that there is an infinite number of integer pairs $n, n+4$, each having exactly one prime factor congruent to 3 (mod 4). (Received July 28, 1941.)

362. Ivan Niven: *Quadratic diophantine equations in the rational and quadratic fields.*

Consider the general quadratic equation in two variables with rational integral coefficients, with non-negative discriminant (this restriction being imposed in order that the graph of the equation be not restricted to a finite region of the plane). Then one solution in integers of this equation implies infinitely many such solutions if and only if the graph of the equation is not an hyperbola with rational asymptotes or a pair of essentially irrational straight lines. If the coefficients of the equation are integers of a real quadratic field, one solution in integers of the field implies an infinite number if and only if the equation does not represent one of the following: no locus, a point, a pair of straight lines with coefficients essentially outside the quadratic field, or an ellipse with totally negative discriminant. A similar theorem is obtained for imaginary quadratic fields, the results being similar to the rational case. The principal parts of these theorems are proved by use of criteria for the solvability of the Pell equation in rational and quadratic fields; the criteria for quadratic fields are determined with the help of a theorem of Hilbert on the units of algebraic fields. (Received July 28, 1941.)

ANALYSIS

363. R. P. Agnew: *Analytic extension by Hausdorff methods.*

Let $\chi(t)$, $0 \leq t \leq 1$, be a mass function which generates a regular Hausdorff method of summability $H(\chi)$. Let the number r , which is the greatest lower bound of numbers ρ such that $\chi(t)$ is constant over $\rho \leq t \leq 1$, be called the *order* of $H(\chi)$. Let $\sum c_n z^n$ be a power series with a positive finite radius of convergence. Corresponding to each vertex ζ of the Mittag-Leffler star, let $B(r, \zeta)$ denote the set of points z for which $|z - (1-r^{-1})\zeta| < r^{-1}|\zeta|$. Let $B(r)$ denote the set of inner points of the intersection of the sets $B(r, \zeta)$. It is shown that $\sum c_n z^n$ is uniformly summable $H(\chi)$ over each closed subset F of $B(r)$. The geometric series $\sum z^n$ is non-summable $H(\chi)$ at each point exterior to the closure of $B(r)$. A series $\sum u_n$ is called summable $\mathcal{H}$ to σ if there is at least one regular method $H(\chi)$ which evaluates $\sum u_n$ to σ . Some properties of the method $\mathcal{H}$ are obtained. The existence of series with bounded partial sums which are not summable $\mathcal{H}$ is implied by the following Tauberian gap theorem. If $0 < n_1 < n_2 < \dots$, if $n_{p+1}/n_p \rightarrow \infty$ as $p \rightarrow \infty$, if $u_n = 0$ when $n \neq n_1, n_2, \dots$, and if $\sum u_n$ is summable $\mathcal{H}$, then $\sum u_n$ is convergent. (Received June 23, 1941.)

364. R. P. Agnew: *On Hurwitz-Silverman-Hausdorff methods of summability.*

Let $H(x)$ denote a regular transformation of the type defined by Hurwitz and Silverman and by Hausdorff, and let $\mu(z)$ be the moment function generated by $\chi(t)$. It is shown that if q is a complex number for which $q \neq 0$ and $\Re q \geq 0$, then the sequence $k!/(k-q)!$ is summable $H(x)$ if and only if $\mu(q) = 0$. Applications are given to the problems of relative inclusion of methods $H(x)$, inclusion of methods $H(x)$ by Abel's method, and omission and adjunction of terms in summable series. (Received July 25, 1941.)

365. R. P. Agnew and Mark Kac: *On differences of elements of sets having positive Haar measure.*

Let $|A|$ denote the Haar measure of a set A in a metric separable locally compact space E whose elements x constitute a continuous group. Using $+$ for the symbol of combination, let the element 0 for which $x+0=0+x=x$ be the origin of the space, and let $-x$ denote the inverse of x . It is shown that if $|A| > 0$, then the set $\Delta(A)$ of points x representable in the form $a-b$ where $a, b \in A$ contains an open set containing the origin. Applications and related results are given. (Received July 25, 1941.)

366. J. D. Bankier and Walter Leighton: *Arithmetic continued fractions.*

Galois' theorem is generalized. A convergent pure periodic continued fraction, none of whose partial numerators is equal to zero, is considered. The value y of the continued fraction is determined by solving a quadratic equation, and the other root z is called the conjugate of y . It is shown that if the continued fraction which is formed by reversing the period converges, its value is $-z$. This result is used to study proper continued fractions. It is proved that, if $y = w + I$, where w is a reduced quadratic irrational with an odd number of terms in the period of its regular continued fraction expansion, and I is an integer, then y admits an infinitude of periodic proper continued fraction expansions, the partial numerators being equal to the same positive integer and the period consisting of two terms with a preliminary period of two terms. Necessary and sufficient conditions that a fraction be equal to an approximant of a proper continued fraction are obtained. (Received July 28, 1941.)

367. E. F. Beckenbach: *Vector formulations of Morera's theorem.*

Integral conditions are given which imply that a continuous vector of components $X_i(x_1, x_2, \dots, x_n)$, $i=1, 2, \dots, n$, be both irrotational and solenoidal. For $n=2$, the results reduce to Morera's theorem. (Received August 1, 1941.)

368. Stefan Bergman and D. C. Spencer: *Some properties of pseudo-conformal transformations in the neighborhood of boundary points.*

In this paper the theorem of Lindelöf on the behavior of an analytic function in the neighborhood of a discontinuity is generalized to a pair of functions of two complex variables. (Received July 28, 1941.)

369. D. G. Bourgin: *Some properties of Banach spaces.*

The interrelations of various characterizations of Banach spaces and their subsets are considered. The main concepts are those of the weak topologies, weak compactness and completeness of order $\aleph_\alpha$, a generalization of quasi-uniformity of convergence, weak convergences, an extension of the Helly property to non-denumerable sets of equations and reflexivity. An example is given of a bicomact set in the unit sphere of the conjugate space, E^* , using the weak topology of E^* as functionals, which contain no non-trivial weakly convergent sequences. (Received July 28, 1941.)

370. R. H. Cameron: *Quadratic convolution equations.*

This paper deals with the second degree Stieltjes integral equation $\iint A(x-t) - u)df(t)df(u) + 2\int B(x-t)df(t) + C(x) = 0$, in which $A(x)$, $B(x)$, $C(x)$ are given functions of bounded variation on the infinite interval, and seeks solutions $f(x)$ of the same sort. (All integrals and sums are from $-\infty$ to ∞ .) Sufficient conditions are given for the existence of two and only two normalized solutions, and also for the reality of the solutions. The special cases $\iint P(x-t-u)\phi(t)\phi(u)dudt + 2\int [p_0\phi(t) + Q(t)]\phi(x-t)dt + 2q_0\phi(x) + R(x) = 0$ and $\sum \sum a_{k-\mu-\nu}u_\mu u_\nu + 2\sum b_{k-\mu}u_\mu + c_k = 0$ are also considered, functions and sequences being of class L_1 and l_1 respectively. (Received July 24, 1941.)

371. C. C. Camp: *A convergence proof involving an inseparable multiple contour integral.*

In removing the restriction assumed in a former paper (American Journal of Mathematics, vol. 60, p. 452), that $a_{ji}(x_j)$ maintain its average value over every subinterval for x_j involved in the auxiliary condition the author treats the analogue of $W_j(v_j)$, which is no longer a function of one complex variable. It is proved, however, that the real parts of the zero-places of $W_j(z_1, z_2)$ for the case $p=2$, ($j=1, 2$), are bounded. The residues of the corresponding Green's system are then worked out according to Poincaré's forms for a simple place (z_1^* , z_2^*) and when W_j has a double root while $W_{j-j} \neq 0$ at the place ($z_1^{(p)}$, $z_2^{(p)}$). The proof of convergence requires two lemmas different from previous ones in that they cannot be written as products of integrals of single variables. The polycylinder interpretation is sufficient here although one might include the more general surfaces recently treated by S. Bergman. Extensions to the case of $p > 2$ and to functions $a_{ji}(x_j)$ which may change sign are indicated. (Received July 18, 1941.)

372. M. M. Day: *Ergodic theorems for abelian semi-groups.*

Improving the methods of an earlier paper (abstract 46-5-257), this paper uses the second conjugate space and its compactness properties to simplify the proof of the ergodic theorem for a bounded abelian semi-group of transformations of a Banach space into itself. Various supplementary results are also given. For example there is a non-trivial invariant set-function defined over all the subsets of each abelian semi-group. This and the main theorem imply that every element of a reflexive space B is ergodic in the sense of Alaoglu and Birkhoff (Annals of Mathematics, (2), vol. 41 (1940), pp. 293-309) and under each bounded abelian semi-group of transformations in B . (Received June 28, 1941.)

373. M. M. Day: *Operations in Banach spaces.*

A theorem of Banach (*Théorie des Opérations Linéaires*, p. 80, Theorem 5) states that if A and B are Banach spaces and if U_n , $n=1, 2, \dots$, are linear operators on A

to B such that $\limsup_n \|U_n(a)\| < \infty$ for each a in A , then $\limsup_n \|U_n\| < \infty$. This relation need not hold if the sequence of integers is replaced by any directed set X and this paper is mainly concerned with the problem of boundedness suggested by this. Characterize the triples $\{A, B, X\}$, where A and B are Banach spaces and X is a directed set, such that the operators U_α on A to B can be chosen so that $\limsup_\alpha \|U_\alpha(a)\| < \infty$ for every a in A while $\limsup_\alpha \|U_\alpha\| = \infty$. Let P be the class of such triples. Typical results are these: (1) $\{A, B, X\} \in P$ for every B if there is a B_0 such that $\{A, B_0, X\} \in P$. (2) If $\{A, B, X\} \in P$ and $Y > X$ in the sense of Tukey (*Convergence and Uniformity in Topology*, Princeton, 1940), then $\{A, B, Y\} \in P$. (3) If $\{A, B, X\} \in P$ and there is a linear operator taking a Banach space A' into all of A , then $\{A', B, X\} \in P$. (4) A is not finite dimensional if and only if X and B exist so that $\{A, B, X\} \in P$. (5) X is essentially sequential if and only if no A and B exist such that $\{A, B, X\} \in P$. (Received June 28, 1941.)

374. M. M. Day: *Reflexivity criteria for a Banach space.*

An elementary proof not involving integration is given for a theorem of Goldstine (Duke Mathematical Journal, vol. 4 (1938)) that weak completeness of a general sort implies reflexivity of a Banach space B . The principal tool is a lemma on the distance from the zero element of B to the intersection of a finite number of hyperplanes in B ; an attempt to use the same criterion for the intersection of $\aleph_\lambda$ hyperplanes ($\lambda \geq 0$) leads to a new criterion for reflexivity of B and to the extension of a number of known conditions for reflexivity of B to conditions for reflexivity of every $\aleph_\lambda$ -separable subspace of B . Many of the results are related to those of Šmulian (Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS, vol. 18 (1938)). (Received July 23, 1941.)

375. R. J. Duffin and J. J. Eachus: *A Paley-Wiener type expansion theorem.*

The following expansion theorem is easy to prove: Let $\{A_{nk}\}$ ($n, k = 1, 2, \dots$) be a matrix of constants such that $|A_{nk}| < a_k$. Let $\{T_k\}$ be a sequence of bounded linear transformations with corresponding bounds $\{t_k\}$. Let $\{f_n\}$ be a complete orthonormal sequence of functions and define $g_n = f_n + \sum_{k=1}^{\infty} A_{nk} T_k f_n$. Then if $\sum_{k=1}^{\infty} a_k t_k < 1$ the sequence of functions $\{g_n\}$ is strongly complete in the sense of Paley and Wiener, *Fourier Transforms in the Complex Domain*, p. 100. The modus operandi of the proof is "the method of separation of variables." The theorem is easy to apply upon observing that multiplication by a bounded function is a bounded linear transformation of Hilbert space. For example, new results are obtained for the non-harmonic Fourier series. Thus the sequence of functions $\{e^{i\lambda_n x}\}$, $n = 0, \pm 1, \dots$ where $\{\lambda_n\}$ is a sequence of complex numbers satisfying $|\lambda_n - n| \leq L$ is strongly complete for $L < \log 2/\pi$. This is clearly a better value than $1/\pi^2$ obtained by Paley and Wiener. The best value is not known but a theorem of Levinson gives an upper limit of $1/4$. (Received July 30, 1941.)

376. R. J. Duffin and A. C. Schaeffer: *Power series with bounded coefficients.*

The following theorem of Szegő is well known: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has only a finite number of different coefficients and if $f(z)$ is continuable beyond the unit circle then

it is a rational function. In this paper the following stronger assertion is proved: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has only a finite number of different coefficients and if $f(z)$ is bounded in some arc of the unit circle, then it is a rational function. Of central importance in the proof is a recent result of Cartwright which states that if $\phi(\omega)$ is an entire function such that $|\phi(\omega)| = O(e^{k|\omega|})$, $k < \pi$, and if $\phi(\omega)$ is bounded for integer values of ω then it is bounded on the entire real axis. Some generalizations of Cartwright's theorem are obtained, and it is shown that these generalizations can be used to obtain gap theorems which overlap those of Paley and Wiener, *Fourier Transforms*, p. 124. (Received July 30, 1941.)

377. J. M. Earl: *Polynomials of interpolation on an infinite interval.*

If $f(x)$ is an arbitrary function of x with upper bound M , then the interpolating polynomial of degree n which assumes the value $f(x)$ for $n+1$ values of x has an upper bound which depends on M and the distribution of these $n+1$ values of x . If the interval on which $f(x)$ is defined is infinite and the manner in which $f(x)$ becomes infinite when x becomes large numerically is suitably restricted, then the distribution of the $n+1$ values of x may be chosen so that an upper bound for the magnitude of the product of a weight function and the interpolating polynomial may be obtained for the whole infinite interval. It is possible in this manner to obtain theorems on the convergence of polynomials of interpolation to a given function on an infinite interval. For example, if $f(x)$ has a p th derivative which satisfies a Lipschitz condition on the doubly infinite interval and the $n+1$ points of interpolation are equally spaced on an interval for which $|x| \leq n^{3/4}$, then the difference of $f(x)$ and the interpolating polynomial, multiplied by the exponential of the negative of x square, does not exceed a constant multiple of $n^q \log n$, where $q = -(p+1)/4$. (Received July 31, 1941.)

378. Paul Erdős and Gabor Szegő: *On a problem of I. Schur.*

Let $n \geq 3$, $-1 \leq x_0 \leq +1$, and let $K_n(x_0)$ denote the class of polynomials $f(x)$ of degree n for which $|f(x)| \leq 1$ in $-1 \leq x \leq +1$ and $f''(x_0) = 0$. The main purpose of this paper is the determination of the maximum m_n of $|f'(x_0)|$ as $f(x) \in K_n(x_0)$ and x_0 runs from -1 to $+1$. It is shown that, for sufficiently large n , this maximum is attained for $x_0 = \pm 1$ and for certain polynomials which were introduced and investigated by G. Zolotareff. Furthermore $\lim_{n \rightarrow \infty} m_n = k^{-2}(1 - E/K)^2 = 0.3124 \dots$ where k^2 is the only root of the transcendental equation $(K - E)^2 + (1 - k^2)K - (1 + k^2)E = 0$, $-1 < k^2 < +1$. Here K and E denote the complete elliptic integrals associated with the modulus k^2 . A brief study of the polynomials of Zolotareff is included. Short proofs of two theorems of Zolotareff are given. (Received August 1, 1941.)

379. L. R. Ford: *The proper fractions.*

The proper fractions are the rational numbers between 0 and 1. The following formula is typical of the results obtained in this paper: $\sum (q^2 \sin p\pi/q)^{-2} = 5/\pi^2 - 1/3$, where the summation extends over all proper fractions p/q . (Received July 28, 1941.)

380. Bernard Friedman: *Fourier coefficients of odd functions.*

Given an odd measurable function $f(x)$ whose absolute value is less than one. Let $\sum b_n \sin nx$ be its Fourier series. Then an n th order determinant is obtained which expresses the necessary and sufficient condition satisfied by the first n coefficients. The method is similar to that of a previous paper (this Bulletin, vol. 47 (1941), pp. 84-92). (Received July 30, 1941.)

381. H. L. Garabedian: *A class of linear integral transformations.*

This paper involves a study of a class of integral (kernel) transformations, first considered by Silverman, which has its analogue in the Hausdorff theory of matrix transformations. In view of certain recent developments in the field of Hausdorff matrix transformations it has been found possible to clarify certain obscurities in Silverman's work, and to make significant extensions of Silverman's results on inclusion and equivalence relations among Hausdorff integral transformations. (Received July 3, 1941.)

382. H. L. Garabedian: *Theorems relating to the Cesàro kernel transformation.*

It is the object of this paper to determine conditions on the kernel $k(s, t)$, of a regular transformation $z(s) = \int_0^1 k(s, t)x(t)dt$, in order that the method of summation defined by the transformation shall include Cesàro summability (C, α) with the kernel $\alpha(1-t/s)^{\alpha-1}/s$, $\alpha > 0$. Two theorems are obtained, corresponding to the two cases in which α is an integer or is not an integer, and examples are given in illustration of them. (Received July 3, 1941.)

383. Abe Gelbart: *On some properties of mapping functions.*

In this paper inequalities are obtained for the coefficients of the mapping function in one complex variable which transforms a given domain, simply- or multiply-connected, into a schlicht domain which omits three points, by using the method of the minimum integral (Bergman, Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 16 (1937), pp. 11-14). Analogous inequalities can be obtained for functions of two complex variables. (Received July 28, 1941.)

384. T. F. Glass and Walter Leighton: *On the convergence of continued fractions.*

Sufficient conditions for the convergence of the continued fraction $K[a_n/1]$ are that $|a_{2n+1}| \leq r$, where r is an arbitrary positive number $\leq \frac{1}{2}$, and that the numbers $a_{2n} = \rho_{2n}e^{i\theta_{2n}}$ lie in a region defined by the relations: $\rho_{2n} \geq 2(1+r)^2[1 - \cos(\theta_{2n} + \alpha_0)]$ ($0 \leq \theta_{2n} \leq \pi - \alpha_0$), $\rho_{2n} \geq 2(1+r)^2(\pi - \alpha_0 \leq \theta_{2n} \leq \pi + \alpha_0)$, $\rho_{2n} \geq 2(1+r)^2[1 - \cos(\theta_{2n} - \alpha_0)]$ ($\pi + \alpha_0 \leq \theta_{2n} \leq 2\pi$), where $\alpha_0 = 2 \arcsin r$. This sharpens a criterion due to Leighton and Wall. (Received July 28, 1941.)

385. P. R. Halmos and John von Neumann: *Operator methods in classical mechanics. II.*

A measure space is *normal* if (1) points have measure zero, (2) the family of all measurable sets contains a Borel field B with a countable number of generators, sets of which "cover" every measurable set and separate every two points, and (3) the range of each real-valued univalent function $f(x)$, measurable (B) , is a Borel set. A measure space is isomorphic to the unit interval if and only if it is normal. A subset E of the unit interval is *absolutely invariant* if for every measure preserving transformation T of the unit interval on itself the symmetric difference of E and TE has measure zero. A measure space satisfying conditions (1) and (2) is isomorphic to an absolutely invariant set if and only if every measure preserving automorphism of the Boolean algebra of measurable sets modulo sets of measure zero is induced by a point transformation. Every measure preserving transformation with discrete spectrum on a normal space is isomorphic to a translation on a compact separable abelian group, and

hence to an isometric transformation. Conversely every isometric ergodic measure preserving transformation of a metric, complete, separable space with a regular measure has discrete spectrum. Similar results hold for measurable flows. (Received July 11, 1941.)

386. F. B. Jones: *Measure and other properties of a Hamel basis.*

A Hamel basis is a set of real numbers $a, b, c, \dots$ such that each real number can be expressed uniquely in the form $\alpha a + \beta b + \gamma c + \dots$, where $\alpha, \beta, \gamma, \dots$ are rational numbers of which only a finite number are different from zero. Certain measure conditions under which a set (of a large class of sets, including all analytical sets) shall contain a Hamel basis are obtained which are both necessary and sufficient. Although no Hamel basis is an analytical set, an example of a Hamel basis which contains an infinite perfect set is given. Some of the rather peculiar properties of this perfect set are pointed out. (Received July 21, 1941.)

387. Mark Kac: *Convergence and divergence of non-harmonic gap series.*

Consider the series $\sum_{n=1}^{\infty} a_n e^{i\lambda_n t}$, where λ_n is a real number satisfying the gap condition $\lambda_{n+1}/\lambda_n > q > 1$ ($n = 1, 2, \dots$). If $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, the first series converges almost everywhere for $-\infty < t < \infty$. If a stronger gap condition is assumed; namely, $\lambda_{n+1}/\lambda_n > q > (5^{1/2} + 1)/2$, then the divergence of $\sum_{n=1}^{\infty} |a_n|^2$ implies the divergence almost everywhere of the first series. The proofs refer to the work of Marcinkiewicz and Zygmund on harmonic gap series. (Received July 26, 1941.)

388. S. Kakutani: *A class of examples of mixing flows*

The following theorem is proved: Let Ω be the space of all sequences $\{x_n\}$ ($n = \dots, -1, 0, 1, \dots$), with $0 \leq x_n \leq 1$, and with the usual independent measure defined multiplicatively in terms of Lebesgue measure on each coordinate. Let T be the transformation on this space taking each point $\omega = \{x_n\}$ into the point $T\omega = \{x_{n+1}\}$ and let T_t be a flow built on this transformation under a function $f(\omega)$ which depends only on a single coordinate: $f(\omega) = f(x_0)$. (For the definition of a flow built under a function see W. Ambrose, *Annals of Mathematics*, July, 1941.) Then T_t is a strong mixing flow unless all values taken by the function $f(\omega)$ (except for a set of measure zero) are integral multiples of some constant c . If all values taken by $f(\omega)$ are integral multiples of some constant c , then the flow has a non-empty point spectrum. The proof of this theorem is accomplished by use of a Tauberian theorem of N. Wiener. (Received July 26, 1941.)

389. S. Kakutani and W. Ambrose: *Structure and continuity of measurable flows.*

Various theorems about the structure of measurable flows are obtained. In particular it is shown that on a measure space (i.e., a space on which a countably additive finite valued measure is defined) satisfying the following two conditions: (1) there exists a (countable) sequence of measurable sets such that for every pair of points some member of the sequence contains one but not both of the points, and (2) the measurable sets are obtained by completing (i.e., by throwing in all subsets of sets of measure zero) a Borel field determined by a countable collection of sets (both of these conditions are clearly satisfied by Lebesgue measure in euclidean space) every measurable flow (other than the identity) is isomorphic to a flow built under a function

(for definition see Ambrose, *Annals of Mathematics*, July, 1941) and also to a continuous flow on a separable metric space with a regular measure. (Received July 26, 1941.)

390. R. B. Kershner: *On the packing of convex regions in the plane.* Preliminary report.

By a packing of a convex region R in the plane is meant a configuration consisting of a finite or infinite collection of congruent images of R placed in a nonoverlapping way upon the euclidean plane. The density of such a packing is defined in an obvious way and represents the proportion of space covered. By the packing constant $\gamma(R)$, for a given convex region R , is meant the least upper bound of the set of densities corresponding to all packings of R . It is shown that there is an absolute constant $k > 0$ such that $\gamma(R) \geq k$ for all convex regions R . The regions (polygons) for which $\gamma(R) = 1$ are determined. As a lemma the area A_n of the minimum n -gon containing a given convex region of area A is discussed and the order of magnitude of $1 - (A_n/A)$ is determined. (Received July 28, 1941.)

391. R. E. Langer: *A theory for ordinary linear differential boundary problems of highly irregular type.*

A general theory for boundary problems, which in matrix notation are $y' = \{\lambda R(x) + Q(x)\}y$, $W_a(\lambda)y(a) + W_b(\lambda)y(b) = 0$, has hitherto been given only for cases classified as regular or mildly irregular. For highly irregular problems, results have been obtained only in markedly special cases (papers by J. W. Hopkins, L. E. Ward, and J. I. Vass), and even then no theorems of a generality comparable with those known for regular problems have been found. In this paper the highly irregular problem is approached by a new method, in which it is imbedded in a continuous aggregate of problems of which all other members are regular. Its expansion theory is thus sought through limiting processes from existing theory. A sub-classification of highly irregular problems is found to be called for. For the expansions associated with problems of the one category (which includes all those for which any discussions are in the literature) the convergence of regular expansions is made to yield an appropriately defined summability, under conditions familiar from the theory of Fourier's series. For problems of the other category the method proves inapplicable, and it seems probable that no expansion properties in the usual sense inhere in these cases. (Received July 26, 1941.)

392. J. D. Mancill: *Multiple integral problems of the calculus of variations with prescribed transversality coefficients.*

Single integral problems of the calculus of variations with prescribed transversality conditions have recently been studied by the author (*American Journal of Mathematics*, vol. 41 (1939), pp. 330-334). In the present paper it is shown that the method used in that study applies with slight modification to multiple integral problems. The method applies with equal facility to parametric and non-parametric integrals of all orders of multiplicity but the case of double integrals is presented in this paper for the sake of simplicity of notation. The problem is stated in parametric form and necessary and sufficient conditions in order that a transversality relation belong to such a problem of the calculus of variations are derived. Finally the most general integrand function of such a problem to which a given transversality relation belongs is determined. (Received June 18, 1941.)

393. Szolem Mandelbrojt: *New proof of the conditions of quasi-analyticity.*

The author gives a very simple proof for the necessity of the well known conditions that a family of functions be quasi-analytic. (Received July 28, 1941.)

394. Ralph Mansfield: *Differential systems involving k -point boundary conditions.*

This paper is concerned with a k -point boundary value problem consisting of a system of ordinary linear differential equations and a set of boundary conditions involving linearly the values of the solutions at interior points as well as at the end points of the interval over which the system is defined. By means of a transformation this boundary problem is reduced to a two-point problem. By this device the fundamental properties of the k -point problem and its adjoint system are readily deduced. The definitions of self-adjoint, definitely self-adjoint, self-conjugate adjoint, and definitely self-conjugate adjoint systems are extended to the k -point boundary problem. In addition, necessary and sufficient conditions that such k -point problems arise from certain problems of the calculus of variations are determined. Applications of this method to the study of boundary problems in the complex domain lead to some of the results previously established by Langer. (Received June 3, 1941.)

395. Herman Meyer: *Polynomial approximations to functions defined on abstract spaces.*

The purpose of this paper is to extend the Weierstrass theorem on polynomial approximations to continuous and differentiable functions to the case where the domain of the functions involved is a Banach space with a basis and the range is a general Banach space. For a continuous function the approximating sequence of polynomials (i.e., polynomials in the sense of Gateaux, Bulletin de la Société Mathématique de France, vol. 47 (1919), p. 73) converges on bounded closed sets, uniformly on compact subsets. For a differentiable function, after some further restrictions on the domain, the differentials of the polynomials also converge to the corresponding differentials of the function. Subject to still further restrictions on the domain, an explicit formula to determine the polynomials is presented. A double sequence of Banach valued polynomials defined over euclidean spaces is set up and a proper diagonal subsequence is selected. Well defined linear transformations of the domain onto certain spaces isomorphic to euclidean spaces then produce the desired approximating sequence. (Received July 25, 1941.)

396. M. G. Moore: *On functions satisfying two functional equations.*

Making a study quite similar to that of doubly periodic analytic functions, an examination is made of the class of analytic functions which satisfy two equations of the type $\sum_{\mu=1}^N c_{\mu} f(x + \alpha_{\mu}) = 0$, where c_{μ} and α_{μ} are complex constants, or, more generally, which satisfy two differential equations of infinite order of certain character. In particular, two equations cannot in general have a common non-trivial solution analytic throughout certain convex regions, although they do have solutions analytic throughout smaller regions. The complete analytic solution of a pair of equations is characterized as is also that of three equations. (Received July 29, 1941.)

397. E. H. Nicholson: *On the degree of approximation in some convergence theorems concerning derivatives of the mapping function in conformal mapping.*

Suppose that the function $w=f(z)$ maps the unit circle conformally on the interior of a closed Jordan curve C . It is known that, under suitable assumptions concerning C , the mapping function $f(z)$ and its derivatives vary continuously in the closed unit circle under a continuous deformation of C . For certain problems it is desirable to know the degree of the variation of $f(z)$ and its derivatives in this dependence. This means: if C_1 and C_2 are closed Jordan curves which "differ" by less than some positive ϵ , and if $f_i(z)$ are the corresponding mapping functions ($f_1(0)=f_2(0)$, $f'_1(0)>0$), $i=1, 2$, what can be asserted regarding an upper bound for $|f_1(z)-f_2(z)|$, $|f'_1(z)-f'_2(z)|, \dots$ in terms of ϵ ? The problem concerning the degree of variation of the function $f(z)$ itself has been investigated by L. Bieberbach, A. R. Marchenko, and A. Markouchevitch. The present paper deals with the corresponding problem for the derivatives of $f(z)$. (Received July 30, 1941.)

398. I. E. Perlin: *A calculus of variations problem with end points as functions of the curve.* Preliminary report.

In this paper the author considers a calculus of variations problem with end conditions functions of the curve. By introducing new variables the author transforms this problem to a Lagrange problem with variable end points. Through the use of a generalized Lindeberg theorem sufficient conditions for the original problem can be shown to be deduced from sufficient conditions for the related Lagrange problem (Received July 30, 1941.)

399. R. S. Phillips: *On weakly compact subsets of a Banach space.* Preliminary report.

A subset G of a Banach space X is called $\aleph_\alpha$ -closed if for every subset $\{x_\tau\} \subset G$ defined on a directed set of cardinal power less than or equal to $\aleph_\alpha$ there exists an $x_0 \in X$ such that $\liminf_\tau x(x_\tau) \leq x(x_0) \leq \limsup_\tau x(x_\tau)$ for every $x \in \bar{X}$, the conjugate space. $\aleph_0$ -closed sets are the usual weakly conditionally compact sets, and some properties are presented. If the unit sphere of X is $\aleph_\alpha$ -closed, the unit sphere of $\bar{X}$ is also $\aleph_\alpha$ -closed. If in addition there exists a set of power $\aleph_\alpha$ total on X , then X is reflexive. This is equivalent to a recent result of Bourgin. A linear transformation on X to Y is called $\aleph_\alpha$ -closed if the image of the unit sphere in X is $\aleph_\alpha$ -closed and there exists a set of power $\aleph_\alpha$ total on the closed linear extension of this image. The adjoint transformation of a $\aleph_\alpha$ -closed transformation is likewise $\aleph_\alpha$ -closed. All weakly completely continuous transformations on L to X are separable valued. Finally, any completely additive set function of strong bounded variation on a sigma field to an "almost" $\aleph_0$ -closed subset of X is the indefinite integral of a Bochner integrable function. (Received July 29, 1941.)

400. Harry Pollard: *Legendre polynomials and the Hausdorff moment problem.*

Among the proofs given by Hausdorff of his classical result on the representation of completely monotonic sequences there is one which makes use of the $(C, 2)$ kernels for Legendre series. In the present paper the theorem in question is deduced from the fact that the Abel kernel for these series is also non-negative. In terms of the Legendre

polynomials, a new criterion is then obtained for the representation of a sequence in the form, $\mu_n = \int_0^1 t^n \phi(t) dt$, with $\phi(t)$ bounded. This yields a simple proof of Steinhaus' result on the form of the general linear functional on the space L . (Received July 31, 1941.)

401. Arthur Rosenthal: *On differentiation of integrals and approximate continuity.*

According to Lebesgue the derivative of an integral is equal to the integrand almost everywhere, and according to Denjoy the integrand (assumed as bounded) is furnished by differentiation certainly wherever it is approximately continuous. While in the generalization of Lebesgue's theorem to more dimensions or to abstract spaces, the system of sets relative to which one differentiates must be rather special, the generalization of Denjoy's theorem can be made by use of quite arbitrary indefinitely fine systems of sets. Under certain conditions the approximate continuity is shown to be necessary and sufficient for differentiation of the integral to furnish the integrand. (Received July 11, 1941.)

402. Raphael Salem: *On sets of multiplicity for trigonometrical series.*

Let P be a symmetrical perfect set constructed on $(0, 2\pi)$ by consecutive trisections, the central intervals being removed, and each of the 2^{p-1} trisections of the p th step being made proportionally to ξ_p , $1-2\xi_p$, ξ_p ($0 < \xi_p \leq \frac{1}{2}$; $p=1, 2, \dots$). Let δ_{pk} ($k=1, 2, \dots, 2^p-1$) be the 2^p-1 intervals removed after the p th step and let $F(x)$ be the singular monotonic function of the Cantor type constructed on P and such that $F(x) = k/2^p$ in δ_{pk} . Let c_n be the Fourier-Stieltjes coefficient of dF in respect of e^{inx} . Let $\{a_k\}$, $\{b_k\}$ be two sequences such that $0 < a_k < b_k < \frac{1}{2}$, and consider all the sets P for which $a_k \leq \xi_k \leq b_k$. Following Steinhaus, the infinitely many dimensional domain $a_k \leq \xi_k \leq b_k$ ($k=1, 2, \dots$) is mapped on the interval $0 \leq t \leq 1$. Supposing that $b_k - a_k \geq 1/\omega(k)$, $\omega(k)$ being increasing and such that $\log \omega(k) = o(k)$, the following theorems are proved: If $\liminf (a_1 a_2 \dots a_p)^{1/p} = \alpha > 0$, then for almost all t , (a) the sets P are sets of multiplicity, (b) $\sum |c_n|^s < \infty$ for $s > s_0 = s_0(\alpha)$, (c) $c_n = o(n^{-\delta})$ for $\delta < \delta_0 = \delta_0(\alpha)$. The result (a) holds on the more general assumption $\liminf p^{1/p-\beta} (a_1 a_2 \dots a_p)^{1/p} > 0$, β being any positive number. (Received July 8, 1941.)

403. D. C. Spencer: *A function-theoretic identity.*

Let $n(r, a)$ be the number of roots of $f=a$ in $|z| < 1$. Suppose that $g(R)$, defined for $R \geq 0$, is absolutely continuous, and that $G(R) = \int g(R) d(\ln R)$. Then for any f regular in $|z| < 1$, we have $r \{d/dr\} \int_0^R G(|f|) d\phi = \int_0^R \int_0^R g'(|f|) \cdot |f|^{-1} \cdot |f'|^2 \rho d\rho d\phi + 2\pi n(r, 0) \cdot g(0)$. Taking $g(R) = 1$, λR^λ ($\lambda > 0$), and $1/(1+R^2)$, we obtain respectively the identities (heretofore considered distinct) of Jensen, Hardy-Stein, and Ahlfors-Shimizu (the latter expressing Nevanlinna's characteristic function $T(r)$ in terms of area on the Riemann sphere). Application is made of the Ahlfors-Shimizu formula to obtain a localization of the well known Nevanlinna theorem that, if $T(r) = O(1)$, then f has non-tangential limits almost everywhere on $z=1$. (Received July 17, 1941.)

404. Otto Szász: *Some new summability methods with applications.*

One associates with a series to function transform (method of convergence factors), or with a sequence to function transform certain triangular matrices, and considers the relations between the corresponding summability methods. Such methods permit interesting applications to Fourier series, in particular in connection with Poisson's and Lebesgue's methods of summability. (Received July 23, 1941.)

405. S. M. Ulam: *On measures for subsets of sets of measure zero.*

Given a class K of sets Z of Lebesgue measure zero, one desires to introduce a (finitely additive) measure $m_Z(X)$ for a class of subsets X of each Z with the following properties: $m_Z(Z) = 1$; $m_{Z+Z'}(X) = m_Z(X) + m_{Z'}(X)$. A construction of such a measure is possible for various classes K of sets Z , in particular for the class of all Borel sets Z of Lebesgue measure zero. This problem is related to the problem of relativization in the theory of probabilities. (Received July 30, 1941.)

406. S. M. Ulam: *Theory of the operation of products of sets. I. Preliminary report.*

The product of two sets A and B is the set $A \times B$ consisting of all couples of elements (a, b) , where $a \in A, b \in B$; $A^* = A \times A, A^n = A \times A^{n-1}$. A systematic investigation of the combinatorial and set-theoretical properties of this operation is undertaken. Systems of sets, closed under the boolean operations and the operation of product, are characterized abstractly. The Borel field over the class of all sets of the form $X \times Y$, where $X \in A$ and $Y \in B$ is studied. The operation α of Suslin on sets of this class is also studied. Among other results the existence of sets of all Borel classes, of analytic sets, of non-analytic sets in the above sense, is established. Two subsets E and F of A^* are called product-isomorphic if there exists a one-to-one transformation T of A into itself such that the transformation $(a_1, \dots, a_n) \rightarrow (T(a_1), \dots, T(a_n))$ of A^n into itself carries the set E into F . The set-theoretical properties of this notion, of importance in applications to topological and algebraical problems, are investigated. A notion of "constructiveness" for abstract algebraical structures is introduced. (For the first report on this subject, see this Bulletin, abstract 44-3-132.) (Received July 30, 1941.)

407. František Wolf: *On generalized boundary values and the uniqueness of corresponding Dirichlet problems.*

If $u(z)$ is harmonic in a domain D , $u^D(\zeta)$ is defined at a frontier point ζ as a many-valued function which takes all the limiting values of $u(z_k)$ for all sequences $z_k \subset D$ and $z_k \rightarrow \zeta$. Similarly $u_D(\zeta)$ takes only those limiting values of $u(z_k)$ for which all z_k lie in a sector of ζ . If a system of curves $L(\zeta)$ is defined such that to every frontier point ζ there is an $L(\zeta)$ lying in the sector of ζ with one endpoint at ζ , then $u_L(\zeta)$ takes all the limiting values of $u(z)$ as $z \rightarrow \zeta$ along $L(\zeta)$. If $u^D(\zeta) = 0$ at all frontier points, then $u(z)$ is continuous in the closed D , and as it is harmonic $u(z) = 0$. If D is the unit circle and $u_D(\zeta) = 0$ at all frontier points and $|u| \leq \exp \{1/(1-r)^m\}$, m arbitrary, then $u(z) = 0$. If D is the unit circle, $L(\zeta)$ the radii, $u_L(\zeta) = 0$ at all frontier points, $|u| \leq \phi/(1-r)^m$, ϕ and m arbitrary, then $u^D(\zeta) = 0$ at all frontier points except for a reducible set of them. The function $u(z)$ is harmonic in the whole plane with the exception of these points. These results hold for more general domains and more general $L(\zeta)$ (F. Wolf, Acta Mathematica, vol. 74 (1941), pp. 65-100). (Received June 12, 1941.)

408. František Wolf: *On the summability of trigonometrical integrals.*

For (C, k) sums of a trigonometrical integral $C_k(x, \omega) = \int_0^\omega (1 - \lambda/\omega)^k (\cos \lambda x dA(\lambda) + \sin \lambda x dB(\lambda))$ which are formally of Stieltjes type, but may be defined by one partial integration, the following results are valid: (A) If (i) $\int_0^\infty (\cos \lambda x dA(\lambda) + \sin \lambda x dB(\lambda)) / \lambda^{k+m} = F(x)$ converges uniformly, (ii) $F(x) = \int_a^b dx \omega^m f(x)$ for $x \in (a, b)$,

(iii) $\int_0^\omega (1-\lambda/\omega)^k dA(\lambda)$, $\int_0^\omega (1-\lambda/\omega)^k dB(\lambda)$ converge as $\omega \rightarrow \infty$, then $C_k(x, \omega)$ equiconverges with the (C, k) means of the Fourier series of $f(x)$ in $(a+\epsilon, b-\epsilon)$, $\epsilon > 0$. At the same time the allied trigonometrical integral $\bar{C}_k(x, \omega)$ equiconverges with the trigonometrical series allied with the Fourier series of $f(x)$ in the same interval. (B) If $\limsup C_k(x, \omega) < \infty$ for $\omega \rightarrow \infty$ in a set of positive measure, then $C_k(x, \omega)$ and $\bar{C}_k(x, \omega)$ are bounded for almost all x of E as functions of ω and if furthermore (i) and (ii) are satisfied, then the result of the preceding theorem holds good. (C) If $|C_m(x, \omega)| \leq \psi(x) \in L$ in (a, b) , then the result of the first theorem holds for any $k > m$. (D) If $\int_a^b |C_m(x, \omega)|^p dx < M$, then the result of the first theorem holds for any $k > m+1/p$. (Received June 23, 1941.)

409. J. W. T. Youngs: *A generalized Lebesgue integral.*

S. Banach has extended the Lebesgue integral to all bounded functions defined on a finite interval (*Théorie des Opérations Linéaires*, Warsaw, 1932). The extended integral enjoys several of the standard properties of the Lebesgue integral, but properties concerned with termwise integrability of a sequence are lacking. It is shown here that if convergence of a sequence of functions is understood to mean convergence in (a general) measure, then the Lemma of Fatou is true; that is, the extended integral is a lower semi-continuous functional on non-negative functions. This fact is used to extend further the integral to unbounded functions. All the properties mentioned above together with the usual theorems on termwise integrability of a sequence hold. Several applications are made. (Received July 24, 1941.)

APPLIED MATHEMATICS

410. E. L. Buell: *On the distribution of plane stress in a semi-infinite plate with partially stiffened edge.* Preliminary report.

The concentration of stress at the straight edge of a thin semi-infinite plate near the point of application of a concentrated shear load acting in the plane of the plate is reduced if the load is applied, not directly to the plate, but to an elastic stiffening rod attached along its edge. When this rod does not extend along the entire edge the boundary value problem of bi-potential theory for the Airy stress function has non-uniform boundary conditions. To solve this problem a conformal mapping of the slit full plane into the interior of the unit circle is employed. A solution of the resulting transformed boundary value problem is obtained in the form of a Fourier series, the coefficients of which satisfy an infinite system of linear equations in an infinite number of unknowns. This system has been solved approximately for the case of a stiffening rod extending to infinity in one direction from the loading point, care being taken to improve the convergence by first separating out the discontinuous parts of the solution. The resultant expressions for those stresses which are of interest have been derived and evaluated numerically. (Received July 28, 1941.)

411. A. S. Householder: *A theory of steady-state activity in nerve-fiber networks. IV: n circuits with a common synapse.*

Let A_i be the product of the activity parameters for the i th circuit (for terminology see Bulletin of Mathematical Biophysics, vol. 3 (1941), pp. 63-69, 105-112). Then an arbitrary stimulus pattern (SP) determines uniquely an activity pattern (AP) if and only if every A_i , as well as the sum of any number of distinct A_i , is less than unity. In case this condition fails only for the sum of all the A_i , then the possible

AP 's can be classified as "even," if it is even with respect to each circuit (see above reference), "odd," if it is odd with respect to each circuit, otherwise "mixed." Then, as for the simple circuit, given any even AP and any odd AP , there exists an SP consistent with both, but no other pairing of distinct mutually consistent AP 's is possible. (Received July 25, 1941.)

412. Wilfred Kaplan: *Topology of the two-body problem.*

The differential equations of the two-body problem, when written as four first order equations, represent a flow in 4-space. If the energy constant (assumed negative) is fixed, the trajectories are restricted to a three-dimensional manifold E . The paper analyzes the topological structure of E and of the family of trajectories in E . It is found in particular that E is homeomorphic with 3-space minus a line and that the family of trajectories is built up of families on tori. (Received July 21, 1941.)

413. A. N. Lowan, Norman Davids, and Arthur Levinson: *Tables for Gauss' mechanical quadrature formula.*

The Mathematical Tables Project described in the May 1941 issue of the Bulletin has prepared a 15 place table of the zeros of the first sixteen Legendre polynomials together with the corresponding weight coefficients in Gauss' mechanical quadrature formula. The computation was carried out by a self-checking iterated process of synthetic division, and further checked by employing various properties of the zeros and the weight coefficients. The present table constitutes a considerable extension of that of Gauss. Several errors were found in the latter. (Received July 3, 1941.)

414. Isaac Opatowski: *On the theory of lethal irradiation of microorganisms. I.*

The theory of irradiation of a homogeneous aggregate of microorganisms is based on the hypothesis that an organism must absorb a certain number $n+1$ of lethal quanta in order to be killed. If $N_i(t)$ is the number of organisms which have absorbed exactly i quanta during a time t ($i=0, 1, \dots, n+1$), then: $dN_i/dt = k_i N_{i-1} - k_{i+1} N_i$, where $k_0 = k_{n+2} = 0$ and all other k_i are positive constants dependent on the vulnerability which an organism has after an absorption of exactly i quanta. The initial conditions are: $N_0(0) = a$ known constant $= N$, $N_i(0) = 0$ for $i \geq 1$. An application of the Laplace transformation gives the number of killed organisms $N_{n+1}(t)$ as the repeated Faltung (G. Doetsch, *Laplace-Transformation*, pp. 157-158): $N k_1 k_2 \dots k_{n+1} \cdot 1 \cdot \exp(-k_1 t) \cdot \exp(-k_2 t) \cdot \dots \cdot \exp(-k_{n+1} t)$, which is a symmetric function of $\{k_i\}$. This symmetry has an important consequence: the counting of killed organisms alone (which is the usual experimental method for the determination of the constants k_i) cannot yield any information as to the variation of the vulnerability of the organism due to the absorption of lethal quanta. (Received July 25, 1941.)

415. Eric Reissner: *On the equations of the stability theory of thin shells.*

Given an equilibrium state of stress in a thin shell such that the stress resultants M and N and the shape r of the deformed middle surface are known. The stability of this state against small disturbances depends on whether or not the linearized homogeneous equations of equilibrium for a shell with stress resultants $M + \delta M$ and $N + \delta N$ and a shape $r + \delta r$ admit solutions such that $M + \delta M$, $N + \delta N$ and $r + \delta r$ satisfy the same boundary conditions as those prescribed for M , N and r . The prob-

lem is thereby reduced to one of the determination of branch points of a system of nonlinear equations by means of a characteristic value problem for a system of linearized equations. The relations between δM , δN and δr are, for an elastic shell, those following from Hooke's law and from Navier's hypothesis. The complete system of scalar equations of this problem is obtained by a method analogous to that employed by the author in his simplified derivation of the equations for small displacements of thin shells (American Journal of Mathematics, vol. 63 (1941), pp. 177-184). (Received July 28, 1941.)

416. Alexander Weinstein: *On the buckling of a rectangular clamped plate compressed in one direction.*

This problem can be solved by an extension of a general method of reduction of eigenvalue problems (A. Weinstein, Mémorial des Sciences Mathématiques, vol. 88 (1937)). It can be linked to the corresponding problem for a supported plate by a sequence of intermediate differential problems of the fourth order which give lower bounds for the eigenvalues. It follows from this method that the lowest eigenvalue for a square plate of area π^2 is >10.0 while an upper bound 10.4 has been computed by J. L. Maubetsch (Journal of Applied Mechanics, vol. 49 (1937)) who used the Rayleigh-Ritz method. The present method which gives definitely lower bounds for all eigenvalues differs essentially from previous formal procedures which could not establish such results. (Received July 18, 1941.)

GEOMETRY

417. E. F. Allen: *On a triangle inscribed in a rectangular hyperbola.*

In the study of inversive geometry the following formulas: $z\bar{z}=a^2$, $a^2z+t_1t_2\bar{z}=a^2(t_1+t_2)$, $a^2z+t_1^2\bar{z}=2a^2t_1$, $\bar{p}z+p\bar{z}=2a^2$ are respectively the self-conjugate equation of a circle, the equation of a line through two points on the circle, the equation of the tangent line, and the equation of the polar line of the point p with respect to the circle, where $z=x+iy$, $\bar{z}=x-iy$, and i is defined by the equation $i^2=-1$. If a point in the xy -plane is designated by $z=x+ry$, $\bar{z}=x-ry$, and r is defined by the equation $r^2=+1$, the base $z\bar{z}=a^2$ is the rectangular hyperbola $x^2-y^2=a^2$. It is proved that the above formulas hold. They still hold if r is defined by $r^2=-k$ or $r^2=+k$, where k is a real number. For any triangle inscribed in a rectangular hyperbola there exists a nine-point hyperbola having many of the characteristics of the nine-point circle of a triangle. An anti-orthocentric group of triangles is defined and it is proved that the four triangles of the group have a common nine-point hyperbola. (Received July 17, 1941.)

418. E. F. Beckenbach: *On the analytic prolongation of a minimal surface.*

In extension of a known result concerning the interior behavior of minimal surfaces, it is shown that if a minimal surface is bounded in part by a plane curve and if the surface approaches the plane orthogonally, then the surface can be extended analytically across the plane and the plane is a plane of symmetry for the extended minimal surface. (Received August 2, 1941.)

419. Nathaniel Coburn: *Semi-analytic unitary subspaces of unitary spaces.*

Suppose a Hermitian space X_m of m -dimensions is imbedded in an n -dimensional

unitary space K_n . By use of composite invariants of K_n , a theory of X_m in K_n is constructed which is a generalization of the known theory for K_m in K_n (analytic unitary subspaces in K_n). This X_m is invariant under the semi-analytic group of subspace coordinate transformations. By definition the X_m becomes an S_m (semi-analytic unitary subspace) when a set of preferred coordinate systems exist, related by a group of coordinate transformations (the group of S_m), for which certain components of the connection of X_m vanish. It is shown that the analytic group is a subgroup of the group of S_m which in turn is a subgroup of the semi-analytic group of X_m . The metric tensor of S_m is discussed and it is shown that if a coordinate system exists in S_m for which the ranks of certain matrices formed from the metric tensor of S_m are m , then S_m is a unitary euclidean space. This surprising result means that the conditions that an Hermitian X_m be an S_m are very restrictive. (Received June 2, 1941.)

420. J. J. DeCicco: *Geometric characterization of functions of n complex variables.*

In this paper Kasner's geometric characterization of the infinite group, consisting of two functions of two complex variables (Transactions of this Society, vol. 48 (1940), pp. 50-62), is generalized from four to $2n$ -dimensional space S_{2n} . The infinite group of all correspondences in S_{2n} consisting of n functions of n complex variables has been termed the pseudo-conformal group G by Kasner. In the present paper it is proved that under G an r dimensional hypersurface and a $(2n-r)$ dimensional hypersurface possess at their point of intersection r (if $r \leq n$) independent invariants. For any fixed value of r this characterizes G . Perhaps the simplest characterization (obtained as a corollary of the preceding by setting $r=1$) is that the pseudo-conformal group G of S_{2n} is defined by the preservation of the pseudo-angle between any curve C and any $(2n-1)$ -dimensional hypersurface H at their common point of intersection. For $n=2$, this theorem becomes Kasner's characterization of the pseudo-conformal group in four dimensions. (Received July 21, 1941.)

421. J. H. Giese: *Conformally flat hypersurfaces of spaces of constant curvature.*

In this paper conformally flat n -dimensional ($n \geq 4$) riemannian spaces that can be imbedded isometrically locally in $(n+1)$ -dimensional spaces of constant curvature are characterized by means of the characteristic roots and elementary divisors of the characteristic matrices of the metric and Ricci or second fundamental tensors. These spaces are intrinsically rigid in the enveloping space. Either all principal normal curvatures are equal, which for simple elementary divisors yields spaces of constant curvature, or else there is an $(n-1)$ -fold principal normal curvature and a simple one. In the latter case there exists a preferred congruence of lines of curvature. This congruence is normal. The corresponding orthogonal $(n-1)$ -dimensional hypersurfaces are of constant curvature and are mapped conformally upon one another by the preferred lines of curvature. The existence of such spaces is shown by constructing all suitable metric tensors $g_{ij} = e_i e_j \delta_{ij}$, each e_i being $+1$ or -1 . These metrics are divided into $n+1$ inequivalent classes, n of which are proved non-null by Liouville's theorem on conformal mapping and by the groups of motions admitted by these g_{ij} 's. (Received July 17, 1941.)

422. F. E. Hohn: *On Cartan's projective involutes.*

In his *Leçons sur la Théorie des Espaces à Connexion Projective*, Cartan defines a

projective involute of the curve (C) , $A(t)$, to be a curve $P(t)$ such that for each t the tangent at P passes through A , and such that P lies on the osculating conic of (C) at A . If this last restriction be removed, the projective involutes of (C) are given by $P = (1 + \theta)A + \theta A' + \theta^2 A''$, where θ is an arbitrary function of t . Some of the systems obtained by putting conditions on θ' are studied, as is also the system of curves $P = A + k(c-t)A' + k(c-t)^2 A''$. Various general cross-ratio properties are established, and Cartan's systems of involutes are developed as special cases. (Received July 28, 1941.)

423. T. R. Hollcroft: *Types of distinct double points of primals.*

There are r types of distinct simple double points of a primal f in S_r . A double point of type k , $1 \leq k \leq r$, of f at a point P has a quadric tangent cone of species k whose vertex is an S_{k-1} tangent to f at P . For $k=1$, the double point is a node. For $1 \leq k \leq r-2$, the cone is not composite. For $k=r-1$, the cone consists of two primes and the point is a binode. When $k=r$, the two primes coincide and the point is a unode. The number and forms of the characteristic conditions for each type of double point are obtained. Equations of primals of order n having any of the above types of double point are written. In particular, the following equation $[\sum_{i=1}^{r-k+1} \lambda_i \phi_i^2 + [\sum_{j=r-k+2}^r \lambda_j \psi_j^2] = 0$, wherein ϕ and ψ are non-singular primals of orders 3α and 2α respectively, defines a system of primals of order 6α which has as basis points $2^{k-1}3^{r-k+1}\alpha^r$ double points of type k , $1 \leq k \leq r$. (Received July 17, 1941.)

424. Edward Kasner and J. J. DeCicco: *Pseudo-conformal geometry: functions of two complex variables.*

Kasner has termed the infinite group G of all correspondences in four-dimensional space S_4 , consisting of pairs of functions of two complex variables, the pseudo-conformal group G . He has proved that G is characterized by the preservation of the pseudo-angle between any curve C and any hypersurface H at their point of intersection (Transactions of this Society, vol. 48 (1940), pp. 50-62). In the present paper are found all the invariants of first order between the curves, surfaces, and hypersurfaces at a fixed point under G . A general pair of curve (hypersurface) elements possesses no invariants, whereas an isoclinal pair has a unique invariant. To any general surface element S there is associated a regulus R of curve elements. There are no invariants between a surface S and a curve element E (hypersurface element π) which is not on (not tangent to) the regulus R of S . On the other hand, if $E(\pi)$ is in R (tangent to R), then there is a unique invariant between $E(\pi)$ and S . Finally two general surface elements possess two independent invariants. (Received July 21, 1941.)

425. Marston Morse and G. A. Hedlund: *Manifolds without conjugate points.*

A directed geodesic g on a two-dimensional Riemannian manifold M is transitive if the elements (an element is a point and a direction at the point) on g are everywhere dense in the totality of elements on M . The existence of transitive geodesics has been proved under certain instability conditions by each of the authors. The hypotheses of all previous proofs have implied that no geodesic have on it two mutually conjugate points. The present paper proves that for a large class of two-dimensional manifolds the hypothesis that no geodesic have on it two mutually conjugate points is sufficient to imply the existence of transitive geodesics. This class includes, in particular, all closed, orientable, class C^2 , two-dimensional Riemannian manifolds of genus greater

than one. The implications of the non-existence of conjugate points are also investigated in the case of surfaces of the topological type of the torus. It is found that to a large extent the geodesics have the properties of those on a manifold for which the Gaussian curvature vanishes identically. If an added hypothesis concerning the non-existence of focal points is invoked, it can be shown that the curvature does vanish identically. (Received June 7, 1941.)

426. Abraham Schwartz: *A consequence of the Ricci equations for Riemannian manifolds.*

The paper considers the normal spaces at a point of a riemannian manifold of m dimensions which is imbedded in a manifold of constant curvature of n dimensions, $V_m \subset S_n$. With each normal space, $\alpha = 1, \dots, k$, there is associated a curvature tensor $H(\alpha)$ in a well known way, each tensor $H(\alpha)$ being related to the preceding one, $H(\alpha-1)$, by a Ricci equation. From these Ricci relationships information concerning the dimensionalities of the various normal spaces can be deduced. For instance, if the first normal space is one-dimensional, then the second normal space is zero or one-dimensional according as the rank of the second fundamental form is ≥ 2 or < 2 ; if the first normal space is two-dimensional, then the second normal space is zero-, one-, or two-dimensional depending on the nature of the elementary divisors of the two second fundamental forms. Analogous theorems are true for the other normal spaces. (Received July 24, 1941.)

427. S. M. Ulam and D. H. Hyers: *On approximate isometries.* Preliminary report.

Let E and F be metric spaces and let ϵ be a positive number. The question studied is the following. If $T(x)$ is a continuous transformation of E into F such that $|\rho(T(x), T(y)) - \rho(x, y)| \leq \epsilon$ for all x and y , then does there exist an isometric transformation $S(x)$ of E into F such that for all x , $\rho(S(x), T(x)) \leq k\epsilon$, where k is a positive constant, depending only on the spaces E and F ? The question is answered in the affirmative for the cases in which E and F are Hilbert spaces or finitely dimensional euclidean spaces. (Received July 30, 1941.)

LOGIC AND FOUNDATIONS

428. Garrett Birkhoff: *Metric foundations of geometry.*

New derivations of fundamental theorems of euclidean, hyperbolic and spherical geometry, with particular reference to the lattice of subspaces, are given. The postulates are: (1) metric postulates of Fréchet, (2) straight line postulates of Menger, in a weakened form valid in any riemannian geometry, (3) the postulate of Pasch. Thus the local compactness postulate of Busemann's system is dispensed with, and his straight line postulates are weakened; on the other hand, the postulate of Pasch is not restricted to 3 points. The proofs are elementary throughout. (Received July 7, 1941.)

429. Henry Blumberg: *A reconsideration of the paradoxes in the theory of sets.*

Proceeding from an intuitive bias that it should be possible to eliminate the paradoxes in question along the line of natural human understanding, without reforming our familiar logic or invoking ingenious technical or professional stratagems, the

author, upon carefully re-examining these paradoxes, has reached the following conclusions: (a) None of them arises if the would-be paradox maker is estopped from employing reasoning that may in fairness be rejected by people of sound understanding. (b) A conception of set may be delineated which accords with natural expectations and by means of which we may build a reliable theory of sets comprehending virtually all the results that have been subjected to attack. The relation is indicated which this conception bears to the basic conceptions, in the matter at issue, of the intuitionists (Brouwer and his school), the formalists (Hilbert and his school), the logicians (Russell and others), and the postulationists (Zermelo, Fraenkel). (Received July 29, 1941.)

430. A. R. Schweitzer: *Concerning general abstract relational spaces.*
III.

On the basis of the general abstract relational space $S_{n+1}(G, H)$ ($n=1, 2, 3, \dots$), $G=H$ =symmetric group on $n+1$ variables and certain axioms elaborating this space, the author constructs axioms for the (finite) algebra of logic analogous to his system K_{n+1} for the foundations of geometry. For $n=3$ the elements of S_4 are $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \omega$ with $\alpha\beta\gamma\delta K$, that is, $\alpha\beta\gamma\delta K \alpha\beta\gamma\delta$. Axioms elaborating S_4 are the following: 1. $\alpha\beta\gamma\delta K \supset \lambda\mu\nu\omega K$. 2. $\alpha\beta\gamma\delta K$ and $\xi \supset \xi\beta\gamma\delta K$ or $\alpha\xi\gamma\delta K$ or $\alpha\beta\xi\delta K$ or $\alpha\beta\gamma\xi K$. 3. $\alpha\beta\gamma\delta K$ and $\lambda\mu\nu\omega K$ and $\xi, \eta \supset \alpha\lambda\xi\eta$ not K , $\beta\mu\xi\eta$ not K , $\gamma\nu\xi\eta$ not K , $\delta\omega\xi\eta$ not K . The complete set of $2^4 K$ tetrads is expressed as a reflexive formal sum $\sum(I)$ and classified into subsums: $\sum(\xi)$ is the sum of all K tetrads containing ξ , and so on. If $\xi\eta\xi\tau K$, then $\sum(\xi\eta\xi\tau) = \xi\eta\xi\tau$. The existence of a unique "empty" sum $\sum(0) = \sum(\alpha\lambda) = \sum(\beta\mu) = \sum(\gamma\nu) = \sum(\delta\omega)$ is assumed. The summands of the various $\sum$'s are replaced by their corresponding expressions in terms of $\sum$ and the $\sum$'s are then represented as products $\sum(\xi\eta) = \sum(\xi) \times \sum(\eta)$, and so on. The preceding continues a paper reported in this Bulletin (abstract 46-9-438). (Received July 22, 1941.)

STATISTICS AND PROBABILITY

431. K. J. Arnold: *On spherical probability distributions.*

Two methods of correspondence for circular distributions to the normal error function have led to non-constant absolutely continuous functions (see F. Zernike, *Handbuch der Physik*, vol. 3, pp. 477-478). The corresponding distributions for the sphere are found. The case of diametrical symmetry for both circle and sphere is discussed. Tables of the probability integrals involved are given and an application in geology is included. (Received July 31, 1941.)

432. I. W. Burr: *Cumulative frequency functions.*

Frequency and probability functions play a fundamental role in statistical theory and practice. They are, however, often inconvenient and difficult to use, since it is necessary to integrate or sum to find the probability for a given range. Theoretically the cumulative or integral frequency function would seem to be better adapted to determining such probabilities, since the latter can be found simply by a subtraction. The aim of this paper is to make a contribution toward the direct use of cumulative frequency functions. Some general properties and theory of cumulative functions are presented with particular emphasis upon certain moment functions adapted to such direct use. Both continuous and discrete cases are included. A list of possible cumulative functions is given and a particular one, $F(x) = 1 - (1+x^2)^{-k-1}$, discussed fully. This function has properties which make it practicable and adaptable to a wide variety

of distribution types. It well illustrates the possibilities of the cumulative approach. (Received July 8, 1941.)

433. J. B. Coleman: *A geometric derivation of Fisher's z-transformation.*

In fitting points in a plane by a line so that the sum of the squares of the perpendicular deviations shall be a minimum, a second line is found for which the sum of the squares of the deviations is a maximum. Let $\sum d^2$ be the sum of the squares of the deviations of the points from the minimum line, and $\sum D^2$ be the sum of the squares from the maximum line. Then $\sum D^2 / \sum d^2 = [(1+r)/(1-r)]$, and $\frac{1}{2} \log (1+r)/(1-r)$ is Fisher's z -transformation for testing the coefficient of correlation. (Received July 21, 1941.)

434. J. H. Curtiss: *On the distribution of the quotient of two chance variables.*

The purpose of this paper is to give an accurate general treatment of the distributions of the quotient and product of two chance variables, with attention first to questions of existence, and then to the derivation of a number of formulas for the frequency functions and distribution functions. The principal formulas derived are (i) a formula for the frequency function of the quotient of two variables with an absolutely continuous joint probability function, (ii) a formula for the distribution function of the quotient of a pair of arbitrary independent variables expressed in terms of the distribution functions of these variables, (iii) a formula similar to (ii), but expressed in terms of the characteristic functions of the variables. Variable distributions are also considered, and a theorem for quotients analogous to the central limit theorem is derived. (Received July 8, 1941.)

435. W. K. Feller: *On the integral equation of the renewal theory.*

As is well known, the equation $U(t) = G(t) + \int_0^t U(t-x) dF(x)$ has frequently been discussed under different forms in connection with the population theory, the theory of industrial replacement, and so on. In the present paper it is shown that, using Tauberian theorems for Laplace integrals, it becomes possible to analyze in detail the asymptotic behavior of $U(t)$ as $t \rightarrow \infty$, and also to solve some other problems which have been discussed in the literature. Strict conditions for the validity of different methods to treat the equation are given together with some modifications found to be necessary. The paper will appear in the Annals of Mathematical Statistics. (Received July 30, 1941.)

436. A. M. Mood: *On the asymptotic distribution of medians of samples from a multivariate population.*

Let two variates x_1 and x_2 have a density function $f(x_1, x_2)$ which, besides being positive or zero and having its integral over the whole space equal to one, shall satisfy these conditions: $\int_{-\infty}^{\infty} f(x_1, 1/n) dx_1 = \int_{-\infty}^{\infty} f(x_1, 0) dx_1 + O(1/n)$, $\int_{-\infty}^{\infty} f(1/n, x_2) dx_2 = \int_{-\infty}^{\infty} f(0, x_2) dx_2 + O(1/n)$. The coordinate system is assumed to have been chosen so that the population median is at the origin. Let $(\bar{x}_1, \bar{x}_2)$ be the median of a sample of $2n+1$ elements drawn from a population with this density function. It is shown that for large samples $(\bar{x}_1, \bar{x}_2)$ is normally distributed to within terms of order $n^{-1/2}$ with zero means and variances and covariances given by certain integrals of $f(x_1, x_2)$. A similar result is true for k as well as two variates. (Received August 2, 1941.)

TOPOLOGY

437. D. G. Bourgin: *Linear topological spaces.*

Most of the results concern convex sets. Some theorems on regularly convex sets and weak closures in vector normed spaces are extended to linear topological spaces (l.t.s.) though in some cases a restriction of local convexity of the l.t.s. is required. As a consequence of these results it is shown that a necessary condition for the metrizable-ability of the conjugate space of the locally convex l.t.s. L is that there exist a denumerable basis of bounded sets such that any bounded set in L is contained in a set of the basis. The sufficiency of this condition is already known and the local convexity requirement may be waived here. (Received July 31, 1941.)

438. S. S. Cairns: *The space of a variable geodesic complex on a sphere.*

A simplicial 2-complex on a sphere, S , will be called *geodesic* if its 2-cells are spherical triangles each of area less than a hemisphere. Consider a geodesic complex τ which covers S . Let $T(\tau)$ be the space each of whose points is a geodesic triangulation of S isomorphic with τ , orientation being preserved, where continuity is defined in terms of the positions of vertices. Theorem. *The space $T(\tau)$ is the topological product of projective 3-space and a $(2n-3)$ -cell, where n is the number of vertices of τ .* The connectedness of $T(\tau)$ implies that, by continuous motions of the vertices and of the geodesic cells which they determine, it is possible to carry τ into an arbitrary isomorphic similarly oriented geodesic complex on S , no 2-cell becoming degenerate during the motion. Let $T_0(\tau)$ be the space defined precisely as above, but with the stipulation that the vertices of some particular 2-cell be self-corresponding under each isomorphism and remain fixed during each motion. Then $T_0(\tau)$ is a $(2n-6)$ -cell. There are ready extensions of these results to geodesic complexes which do not entirely cover S and also to rectilinear complexes on a plane. (Received July 30, 1941.)

439. S. S. Cairns: *Topological mapping of a Brouwer 4-manifold on an analytic Riemannian 4-manifold.*

A *Brouwer n -manifold* is a simplicial complex on which the star of each vertex covers an n -cell and admits a piecewise linear homeomorphic mapping into euclidean n -space. The words "piecewise linear" mean linear on each simplex of the star. Theorem. *Every Brouwer 4-manifold can be homeomorphically mapped onto an analytic Riemannian manifold.* This supplements the results of an earlier paper by the writer (Annals of Mathematics, (2), vol. 41 (1940), p. 796). The deformation problem stated in that paper (p. 808) is involved in the proof of the present theorem. It has now been solved, and the solution is included in an article, not yet published, dealing with variable geodesic complexes on a sphere (abstract 47-9-438). (Received July 30, 1941.)

440. Max Dehn: *On the mapping of closed surfaces.*

A system of curves on a closed surface of genus $p > 1$ is, apart from deformations, characterized by a matrix of integers with $3p-3$ rows and two columns. The mappings of the surface (conserving the indicatrix), apart from deformations, constitute a group Γ_p which can be generated by screwings parallel to $p(3p-1)/2$ curves. The elements of Γ_p may be regarded as transformations of the characterizing matrices. If these matrices are considered modulo 2, then Γ_p degenerates into a finite factor group which

is for $p=2$ the symmetric group of six elements. In consequence of this, an odd number of screwings parallel to non-dissecting curves never yields a deformation. This result and the analogous results for $p>2$ still to be proved, have consequences for the determination of the group Γ_p/k_p , the factor group of the commutator group of Γ_p , which is always a finite cyclic group. For instance it follows that Γ_2/k_2 is the cyclic group of order ten or of order two. (Received July 26, 1941.)

441. F. A. Ficken: *Closure-mappings of partially ordered sets*. Preliminary report.

Closure-mappings of a partially ordered set P onto subsets $f(P)$ are those single-valued mappings $x \rightarrow f(x)$ with the properties, whenever x and y are in P , that: $x \leq f(x)$, $f(f(x)) = f(x)$, and $x \leq y$ implies $f(x) \leq f(y)$. When a set S has a least member $y \geq x$ for each x in P , then and only then is $f(x) = y$ a closure-mapping for which $f(P) = S$. Thus the mapping f and the set $f(P)$ determine each other uniquely, and $f(P)$ either has a least element or has no lower bound. Also associated uniquely with f by means of the equivalence relation $a \sim b$, meaning $f(a) = f(b)$, is a partition of P into sets of a very specific kind. As f varies, the solutions of $f(x) = a$ and of $f(a) = y$ are found to run over convex sets. Subsets S such that $f(S) \subset S$ are discussed. (Received July 30, 1941.)

442. R. H. Fox: *Homotopy type and deformation retraction*.

If $f(X) \subset Y$ and $g(Y) \subset X$ and the map gf is homotopic to the identity, then g is called a left inverse of f and f a right inverse of g . Spaces X and Y belong to the same homotopy type if and only if there is a map $\phi(X) \subset Y$ which has both a right and a left inverse. The mapping cylinder $C_\phi (\supset X + \phi(X))$ is obtained from $X \times [0, 1]$ by identifying $(x, 1)$ and $(x', 1)$ whenever $\phi(x) = \phi(x')$. It is proved that X is a retract of $Y + C_\phi$ if and only if ϕ has a right inverse, and that $Y + C_\phi$ can be deformed into X if and only if ϕ has a left inverse. These theorems constitute an analysis of a theorem proved by J. H. C. Whitehead (Proceedings of the London Mathematical Society, vol. 45 (1939), p. 278). These theorems are generalized and applied to Hopf-Pannwitz deformations and yield a new characterization of closure of a homogeneous n -dimensional polyhedron (Alexandroff and Hopf, *Topologie*, Berlin, 1935, chap. 13, § 4). (Received July 28, 1941.)

443. Orrin Frink: *Topology in lattices*.

Different non-sequential topologies which are definable in terms of the order relation in a lattice are studied and compared. These include the Moore-Smith order convergence of directed sets introduced by Garrett Birkhoff, and the interval topology which results on taking the closed intervals of the lattice as a subbasis for the closed sets of the space. It is shown that a complete lattice is bicomact in its interval topology, but not necessarily in its Moore-Smith topology. Applications are made to lattices which are direct products of chains, to function spaces, and to Boolean algebras. Modifications called the neighborhood topology and the convergence topology are defined for the lattice of all closed sets of a topological space. A topology is introduced for the space whose elements are all continuous transformations of a topological space. (Received June 27, 1941.)

444. O. G. Harrold: *A mapping characterization of Peano spaces*.

Nöbeling (*Reguläre Kurven als Bilder der Kreislinie*, Fundamenta Mathematicae, vol. 20 (1933), pp. 30-46) has established the existence of a continuous mapping of the

circle onto a regular continuum X such that for each point $x \in X$ whose order $o(x)$ is finite, $m(x) \leq o(x)$ holds, where $m(x)$ is the number of times x is covered. The existence of strongly irreducible maps of an interval or circle onto any Peano space containing no free arc has been proved by the author (Duke Mathematical Journal, vol. 6 (1940), pp. 750-752). It is shown in the present paper that if X is a Peano continuum, there is a continuous mapping of the circle onto X such that for any point y in the interior of a free arc, $m(y) \leq 2$. Together with previous results this gives: There is a continuous mapping of the circle onto X such that for any point y of a certain dense subset Y of X , $m(y) \leq 2$. Thus a mapping of minimal character is established. The mapping need not be strongly irreducible or even reducible. (Received July 28, 1941.)

445. E. A. Knoblauch: *Extension of homeomorphisms on the torus and projective plane.*

H. M. Gehman (Transactions of this Society, vol. 28, p. 252) showed that a homeomorphism between two Peanian continua lying in planes could be extended to the planes if the homeomorphism preserved sides of arcs. V. Addisson and S. MacLane (Duke Mathematical Journal, vol. 6, p. 216) showed that a homeomorphism between two Peanian continua lying in spheres could be extended to the spheres if the homeomorphism preserved the relative sense of every pair of triods. In this paper a definition of arc crossings is introduced and it is shown that preservation of arc crossings is equivalent to the preservation of relative sense of triods (on a sphere). Further, let K_i ($i=1, 2$) be a Peanian continuum on a projective plane P_i (or torus T_i). Let ϕ be a homeomorphism of K_1 on K_2 . It is shown that if ϕ preserves arc crossings and bounding simple closed curves that ϕ can be extended to a homeomorphism θ of P_1 on P_2 (or of T_1 on T_2), provided that if K_1 contains no non-bounding simple closed curve and $x \in K_1 - C_1$ (C_1 a bounding simple closed curve of P_1 (or of T_1)) then $\phi(x)$ belongs to a 2-cell on P_2 (or T_2) bounded by $\phi(C_1)$ if and only if x belongs to a 2-cell on P_1 (or T_1) bounded by C_1 . (Received June 7, 1941.)

446. Horace Komm: *Partial orders in euclidean spaces.*

This paper investigates certain questions raised by the definition of the dimension of a partial order (Dushnik and Miller, American Journal of Mathematics, vol. 63 (1941), p. 601). Two partial orders P_n and P'_n are defined on E_n as follows: (1) If a and b are points of E_n , then $a < b$ in P_n if and only if every coordinate of a is less than the corresponding coordinate of b . (2) $a < b$ in P'_n if and only if every coordinate of a is less than or equal to the corresponding coordinate of b , and at least one coordinate of a is less than the corresponding coordinate of b . It is shown that dimension $P_n = \text{dimension } P'_n = n$, for every finite n . The λ -dimension of a partial order P is defined just as the dimension of P except that each linear extension is required to be similar to a subset of the linear continuum. A necessary and sufficient condition that P have λ -dimension is obtained. It is shown that P_n and P'_n have λ -dimension for every finite n , and that λ -dimension $P_n = c$, while λ -dimension $P'_n = \aleph_0$, for every n . (Received July 29, 1941.)

447. J. P. LaSalle: *A characterization of topological spaces.*

A characterization of topological spaces can be given in terms of a non-negative real-valued function defined for each element of a space and each element of a partially ordered set. The partially ordered set contains subsets which are directed systems (strongly partially ordered sets) with properties in relation to the real-valued func-

tion. This "pseudo-norm" is a generalization of Hyers' pseudo-norm. (Duke Mathematical Journal, vol. 5 (1939), pp. 628-634.) In some cases the "pseudo-norm" may seem to be trivial in that it can be merely another notation for set inclusion, though the value of this characterization lies in the fact that it takes as the basic concept in topology the extension of a notion which is a familiar one, namely, that of a norm. The author has found this postulation basis convenient for some purposes, and certain topological properties imply interesting conditions on the "pseudo-norm." The converse of the latter statement is also true. The neighborhood system generated by the "pseudo-norm" is not necessarily a complete neighborhood system, but there exists always a complete neighborhood system to which it is equivalent. (Received June 18, 1941.)

448. R. G. Lubben: *Decompositions of point sets and of portions of spaces.*

A T -normal point set is a closed set of T which can be separated by mutually exclusive open sets from any closed set of T not intersecting it. (1) The Hausdorff decompositions of the perfectly compact space H of Fréchet, T , are characterized by the property of being upper semi-continuous and of having T -normal elements. (2) Let T be the space of the atomic ideal points of the completely regular Hausdorff space S . (For definitions see Transactions of this Society, vol. 49 (1941), pp. 410-466.) (a) An amalgamation point of S is regular if and only if it can be decomposed into a T -normal point set. (b) A decomposition of the maximal S -portion into ideal points is an element of $\delta(S)$ if and only if its elements are decomposable into subsets of T whose totality form a Hausdorff decomposition of T ; $\omega(S)$ gives the finest of these decompositions. (c) An example answers in the negative the question of the author concerning the identity of the atomic regular and the atomic mapping points, as well as Alexandroff's special case concerning his regular and completely regular ends (Matematicheskii Sbornik, vol. 5 (47) (1939), pp. 403-423). (Received July 30, 1941.)

449. A. W. Tucker: *A boundary-value theorem for harmonic tensors.*

In a 3-dimensional euclidean region, bounded by regular surfaces and R_1+1 -ply connected, a harmonic vector v (i.e., a vector field v such that $\text{curl } v = \text{div } v = 0$) is uniquely determined when one assigns (1) the normal component of v on the boundary and (2) the periods of the line-integral of v along R_1 independent 1-cycles (i.e. circuits). This is a vector formulation of a proposition of Lord Kelvin on cyclic irrotational motion of an incompressible fluid. It has the following tensor generalization. In an n -dimensional orientable (positive-definite) Riemannian manifold, with regular boundary and p -dimensional Betti number R_p , a harmonic alternating p -tensor A is uniquely determined when one assigns (1) the boundary value of the dual $n-p$ -tensor of A and (2) the periods of the p -fold integral of A over R_p independent p -cycles. If the periods are all zero, A is the alternating derivative of a " $p-1$ -tensor potential" which furnishes a solution to an analogue of the classical problem of Neumann. The proof of the general theorem rests on an extension of the multiple-integral theory of de Rham and Hodge through the use of relative methods of the sort introduced by Lefschetz to obtain duality and fixed-point theorems for non-closed manifolds. (Received July 12, 1941.)

450. A. D. Wallace: *Regularly ordered systems of sets.*

If X, Y are subsets of a topological space the symbol $X \leq Y$ means $X = Y$ or

$\overline{X} \subset \text{int } Y$. A system $[X]$ is regularly ordered if for each pair X_1, X_2 either $X_1 \leq X_2$ or $X_2 \leq X_1$. Regular systems are self-dual under complementation. These systems are useful in the structural analysis of spaces. The same concept may be applied to certain families of subsets. With the aid of this notion it is possible to extend to compact (=bicomact) connected T_1 -spaces the well known proposition on the existence of non-cutpoints. The paper also proves the existence of minimal continua satisfying certain conditions (under the assumption of normality) and gives a generalization of certain theorems of Janiszewski, Mullikin, Moore and others. (Received July 2, 1941.)

451. G. S. Young: *On the outer boundaries of certain connected domains.*

A study is made of the properties of the outer boundaries of connected domains D such that $\overline{D}$ is a continuous curve (not necessarily compact) in a space S satisfying Axioms 0-5 of R. L. Moore's *Foundations of Point Set Theory*, particular attention being paid to the resemblances between them and simple closed curves. Typical results are: (1) If B is an outer boundary of such a domain D , then every point of B is arc-wise accessible from the component of $S-B$ that contains D ; (2) $S-B$ is the sum of two mutually separated connected domains; (3) No subcontinuum K of B contains more than two limit points of $B-K$; (4) If the set H irreducibly separates S and D is a component of $S-H$, then $D+H$ is a continuous curve if and only if every closed and compact proper subset K of H lies in an arc which is a subset of $D+K$. Necessary and sufficient conditions for local compactness, complete separability, topological flatness, and so on, of B or subsets of B are also given. (Received July 28, 1941.)

452. G. S. Young: *On topologically flat subsets of certain spaces of R. L. Moore.*

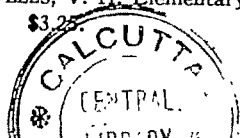
In a space S satisfying Axioms 0-5 of R. L. Moore's *Foundations of Point Set Theory*, a point set M is said to be topologically flat provided that if P is a point and R is a region containing P there is a simple domain D containing P such that the boundary of D plus the common part of D and M is a subset of R . If S is topologically flat in this sense, it is topologically flat in the sense of F. B. Jones (Transactions of this Society, vol. 42 (1937), pp. 53-93). The principal results are: (1) if S is completely separable, it may be imbedded in a completely separable space S' also satisfying Axioms 0-5 in which every flat point set is compact; (2) there is a continuous and reversible transformation which throws S' into a sphere and which is reversibly continuous on every flat subset of S' . Use of these two theorems permits immediate generalization of many plane theorems by omission or weakening of conditions of compactness or connectivity. (Received July 28, 1941.)

453. J. W. T. Youngs: *Concerning arc-curves and basic sets.*

W. L. Ayres has introduced the terms "arc-curve" and "basic set" to the theory of continuous curves (Transactions of this Society, vol. 30, pp. 567-578). In this paper an abstraction is made from the concept of arc-curve upon noticing the fact that if $\overline{X}$ is the arc-curve of X , then: (1) $X \cdot Y \neq \emptyset$ implies that $\overline{X+Y} = \overline{X} + \overline{Y}$, (2) $\overline{X} = X$ if X is empty or a single point, (3) $\overline{\overline{X}} = \overline{X}$. The similarity to the Kuratowski closure axioms is obvious and is extensively used. For example, the arc-curve of a set is comparable to the closure of a set, while a basic set is analogous to an everywhere dense set of points. (Received July 24, 1941.)

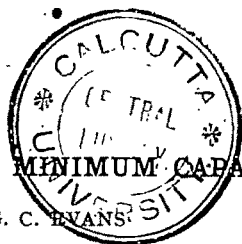
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CONTINUA OF MINIMUM CAPACITY¹

G. C. EVANS



1. **Surfaces containing a given volume.** In an endeavour to simplify a proof of Liapounoff [2], to the effect that in the problem of the forms of equilibrium of rotating liquids the sphere would be the only form for a liquid at rest, Poincaré [1] was led to the consideration of electric capacities of solids of given volume, and arrived at the result that among such bodies the sphere would have minimum capacity. The present paper originated in the question of the determination of the surface sheet, without volume, which would be bounded by a given closed curve in space, and, among all such surfaces, have minimum capacity.

In the discussion of his problem, Poincaré assumes tacitly that there do exist one or more bodies of the given volume, with smooth boundaries, which furnish relative minima for the capacity with respect to neighboring forms; and his treatment amounts to a proof that among these the sphere furnishes the absolute minimum.

Let the body F , with smooth exterior boundary S , be considered as a conductor on which a positive charge m is spread so as to be in equilibrium—that is, the charge lies entirely on the surface S with a surface density $\sigma(P)$, and its potential $V(M)$ has a constant value V_0 within S , is continuous across S , satisfies Laplace's equation $\nabla^2 V = 0$ outside S and vanishes at infinity. The density on S is given by the equation

$$\sigma(P) = -\frac{1}{4\pi} \frac{dV}{dn}, \quad n \text{ the exterior normal.}$$

The energy of the distribution may be written in the equivalent forms

$$(1) \quad I = \frac{1}{8\pi} \int_W (\nabla V)^2 dP = \frac{1}{8\pi} \int_W (V_x^2 + V_y^2 + V_z^2) dP, \quad W = \text{entire space,}$$

$$(2) \quad I = \frac{1}{2} \int_S \sigma(P) V_0 dP = \frac{1}{2} m V_0.$$

The capacity of K of F is defined by the equation

$$(3) \quad KV_0 = m \quad \text{or} \quad I = \frac{m^2}{2K} = \frac{1}{2} KV_0^2.$$

¹ Presidential address presented to the Society, January 1, 1941, under the title *Surfaces of minimum capacity*.

According to Poincaré, the variation δI of I due to a displacement of the points of S , along normals, of amount $\delta n(P) = \epsilon \psi(P)$ is given by the equation

$$(4) \quad \delta I = -\frac{1}{8\pi} \int_S \delta n(P) (\nabla V)^2 dP = -\frac{\epsilon}{8\pi} \int_S \psi(P) \left(\frac{dV}{dn} \right)^2 dP,$$

the integral of the variation of the function $V(P)$ itself being negligible since the conductor distribution is in equilibrium. That (4) thus follows from (1) is not altogether evident. It may be justified, however, in the following way.

Let S_1 be the displaced surface, P_1, Q_1 points on S_1 corresponding to points P, Q on S . Since for a given total mass m on S the energy for a conductor distribution is a minimum, I will have a stationary value for small changes of the distribution $m(e)$ of m , and we may neglect such changes. Hence we may assume the relation

$$\Delta I = \frac{1}{2} \int_S dm(e_Q) \int_S \left(\frac{1}{P_1 Q_1} - \frac{1}{PQ} \right) dm(e_P),$$

the potential at an arbitrary point M being given by the formula

$$V(M) = \int_S \frac{\sigma(P)}{MP} dP = \int_S \frac{dm(e_P)}{MP}.$$

We substitute

$$\frac{1}{P_1 Q_1} - \frac{1}{PQ} = \left(\frac{1}{P_1 Q_1} - \frac{1}{P_1 Q} \right) + \left(\frac{1}{P_1 Q} - \frac{1}{PQ} \right)$$

and change the order of integration in the integral of this second parenthesis, at the same time interchanging P with Q and P_1 with Q_1 , so that we shall have

$$\Delta I = \frac{1}{2} \int_S \sigma(Q) dQ \int_S \left\{ \left(\frac{1}{P_1 Q_1} - \frac{1}{P_1 Q} \right) + \left(\frac{1}{PQ_1} - \frac{1}{PQ} \right) \right\} \sigma(P) dP.$$

We denote by $V_1(Q')$ the potential of the displaced mass at some point Q' between Q and Q_1 , and by $V(Q'')$ the potential of the original mass at some point Q'' between Q and Q_1 . Then

$$\Delta I = \frac{\epsilon}{2} \int_S \sigma(Q) \psi(Q) \left\{ \left(\frac{dV_1}{dn} \right)_{Q'} + \left(\frac{dV}{dn} \right)_{Q''} \right\} dQ.$$

We assume that S and S_1 are sufficiently smooth so that $\sigma(Q)$ is bounded and so that the derivatives of the potential approach the

limits given by the usual well known formulae, as ϵ tends to zero. Then

$$(5) \quad \lim_{\epsilon \rightarrow 0} \left(\frac{dV}{d\mathbf{n}} \right)_{Q''} = \mp 2\pi\sigma(Q) + \int_S \frac{\cos(\mathbf{n}_Q, QP)}{PQ^2} \sigma(P) dP,$$

$$\lim_{\epsilon \rightarrow 0} \left(\frac{dV_1}{d\mathbf{n}} \right)_Q = \pm 2\pi\sigma(Q) + \int_S \frac{\cos(\mathbf{n}_Q, QP)}{PQ^2} \sigma(P) dP,$$

the upper sign being employed when $\psi(Q) > 0$, the lower when $\psi(Q) < 0$. Hence, for $\delta I = \epsilon \lim_{\epsilon \rightarrow 0} (\Delta I / \epsilon)$, we have the value

$$(6) \quad \delta I = \epsilon \int_S \sigma(Q) \psi(Q) dQ \int_S \frac{\cos(\mathbf{n}_Q, QP)}{PQ^2} \sigma(P) dP.$$

Inside the conductor the potential is constant, so that as a point approaches S , from inside, the value $dV/d\mathbf{n}_-$ is zero. Accordingly, by combining the first two of equations (5), we have

$$\frac{dV}{d\mathbf{n}_+} = -4\pi\sigma(Q) = 2 \int_S \frac{\cos(\mathbf{n}_Q, QP)}{PQ^2} \sigma(P) dP.$$

Thus the formula (6) for δI reduces to the value given by (4).

In order that the capacity be a minimum, with given volume, it is necessary that $\delta I = 0$, where δI is given by (4) and $\psi(P)$ is subject to the condition

$$\int_S \psi(P) dP = 0.$$

It follows therefore that $(dV/d\mathbf{n}_+)^2$ is constant on S , and since $dV/d\mathbf{n}_+ = -4\pi\sigma$, we have

$$(7) \quad \sigma(P) = \text{const. on } S.$$

Moreover, since now σ is constant on S , we may write the formula for δI in the form (the total mass m of course remaining unchanged)

$$\delta I = -\frac{S^2 \sigma^2}{2K^2} dK = -2\pi \sigma^2 d\tau$$

in which the volume is changed, and $d\tau$ and dK are corresponding differential changes of volume and capacity. For such changes, then,

$$\frac{S^2 dK}{K^2} = 4\pi d\tau.$$

Suppose now that the body is deformed so as to remain similar to itself. Its capacity will change in the ratio of corresponding lengths, so that

$$\frac{dK}{K} = \frac{1}{3} \frac{d\tau}{\tau},$$

and, in the case of bodies for which (7) is satisfied,

$$K = \frac{1}{12\pi} \frac{S^2}{\tau}.$$

The sphere satisfies the condition (7), and it has the smallest S for a given τ . Hence if there is any smooth S which has the minimum capacity for a given volume, the sphere is that body.

Incidentally, we notice that if we make no requirement about the volume, but consider the surface merely as a sheet or cap, bounded, for instance, by a given closed curve in space, the requirement that the capacity be a minimum implies, if S be sufficiently smooth, that δI be zero, as given by (6). Since there is no restriction on $\psi(Q)$ in the neighborhood of any point of S , not on the boundary, we have the relation

$$(8) \quad \int_S \frac{\cos(\mathbf{n}_Q, QP)}{PQ^2} \sigma(P) dP = 0 = \int_S \frac{\cos(\mathbf{n}_Q, QP)}{PQ^2} dm(e_P).$$

By means of the first of (5) the equation (8) may be written in the form

$$(8') \quad \frac{dV}{dn_+} + \frac{dV}{dn_-} = 0,$$

where, of course, we still have the relation

$$(8'') \quad \frac{dV}{dn_+} - \frac{dV}{dn_-} = -4\pi\sigma.$$

2. Capacity and logarithmic potential. In the discussion of the stability of the form of a rotating liquid, and in particular the spherical form of a liquid at rest, Liapounoff treated the second variation of the energy integral and showed it to be positive. Poincaré endeavored, as we have mentioned, to provide a simpler treatment for this special case. But in order to complete the problem in a satisfactory manner without assuming the existence of a smooth form for which capacity is a minimum further details are necessary. The possibility of a fuller

use of fundamental inequalities for this purpose was noticed by Faber [3] and Szegő [5]. This use is sufficiently illustrated in some corresponding, but perhaps more difficult, problems in the plane, and accordingly we turn to those.

For the plane, as in space, the capacity is to be regarded as an index of the charge which is necessary in order to raise the body to a given constant potential V_0 . The equation $V_0 = m/K$, however, if carried over to the logarithmic potential has the disadvantage of yielding infinite and negative capacities. A more satisfactory definition, especially from the point of view of the complex variable, is given by

$$(9) \quad V_0 = m \log \frac{1}{K}; \quad K = e^{-V_0/m}.$$

This is the one which we adopt. It is equivalent to the "transfinite diameter." By means of it, capacity is of dimension one in length in the plane (as in space), with regard to transformations of similarity, and the capacity of a circle (like that of a sphere) is equal to its radius.

Given an arbitrary bounded closed set F in the plane it may not be possible to distribute the mass m on F in such a way that the potential will remain constant everywhere on F because the frontier s which bounds the infinite domain T exterior to F may not consist entirely of points which are regular, with respect to T , for the Dirichlet problem with continuously given boundary values on s —for example, if s has isolated points. We can however approximate to T by a nested sequence of domains the boundary of each of which consists of a finite number of closed simple analytic curves, and thus define K by the obvious limiting process. In particular, if for a given m the potential V_0 does not remain bounded in the limit, the capacity of F is zero.

Accordingly we may limit ourselves to the sets F whose exterior frontiers consist only of points regular with respect to the Dirichlet problem. Let $u(M)$ be the potential of unit mass on F distributed so as to have constant potential V_0 on F . Then, with $r = OM$ where O is any fixed point,

$$u(M) = \log \frac{1}{r} + H(M) = -\log r + H(M)$$

with $H(M)$ a function which is harmonic in T and vanishes at infinity. Consequently the function (a special case of the Green's function)

$$V(M) = V_0 - u(M),$$

becomes infinite like $\log r$ at ∞ and vanishes on F . In fact,

$$(10) \quad V(M) = \log r - G(M), \quad G(M) = H(M) - V_0,$$

where $G(M)$ is harmonic in T , and regular at ∞ , at which point it takes on the value $G(\infty) = -V_0$. We have therefore

$$(11) \quad K = e^{G(\infty)}.$$

We follow now the procedure as sketched briefly by Szegő for analogous problems [5]. Consider the level curves $V(M) = \lambda$ of the function $V(M)$. For $\lambda > 0$ these curves have no points in common with F since the points of the boundary s are regular; except for a finite number of values of λ , provided that λ is bounded away from 0, the locus $V(M) = \lambda$ consists of a finite number of simple closed analytic curves [6, b], and for large values of λ of a single piece. We denote this locus (and also its total length) by C_λ , the total area inside it by A_λ and the exterior normal direction by $\mathbf{n}$. Thus $\lim_{\lambda \rightarrow 0} A_\lambda = \text{meas. } F$.

We have

$$2\pi = \int_{C_\lambda} \frac{dV}{d\mathbf{n}} ds, \quad \frac{dA_\lambda}{d\lambda} = \int_{C_\lambda} \frac{d\mathbf{n}}{dV} ds,$$

the first equality being an expression of the fact that V is harmonic except for the additive term $\log r$. But by Schwarz's inequality and the isoperimetric inequality applied successively, we have

$$(12) \quad \begin{aligned} 2\pi \frac{dA_\lambda}{d\lambda} &= \int_{C_\lambda} \left(\left(\frac{dV}{d\mathbf{n}} \right)^{1/2} \right)^2 ds \cdot \int_{C_\lambda} \left(\left(\frac{d\mathbf{n}}{dV} \right)^{1/2} \right)^2 ds \\ &\geq \left(\int_{C_\lambda} ds \right)^2 = C_\lambda^2 \geq 4\pi A_\lambda. \end{aligned}$$

Thus $(1/A)dA/d\lambda \geq 2$, and

$$\log A_\lambda - \log A_{\lambda_1} \geq 2(\lambda - \lambda_1).$$

Hence if we let $\lambda_1 \rightarrow 0$, we have

$$A_\lambda \geq (\text{meas. } F) e^{2\lambda}.$$

In this expression we now let $\lambda \rightarrow \infty$. For large values of λ we may use polar coordinates, and write, since $r = e^\lambda e^{G(M)}$,

$$A_\lambda = \frac{1}{2} \int_0^{2\pi} r^2 d\theta = \frac{1}{2} e^{2\lambda} \int_0^{2\pi} e^{2G(M)} d\theta.$$

Consequently

$$\frac{1}{2} \int_0^{2\pi} e^{2G(M)} d\theta \geq \text{meas. } F;$$

and if we let $\lambda \rightarrow \infty$ so that $G(M) \rightarrow G(\infty)$, we shall have

$$\lim_{\lambda \rightarrow \infty} \int_0^{2\pi} e^{2G(M)} d\theta = 2\pi K^2.$$

Hence

$$(13) \quad \pi K^2 \geq \text{meas. } F.$$

The equality sign in (12) can hold only if it holds identically in λ ; that is, by Schwarz's inequality, only if dV/dn and dn/dV are proportional on each C_λ so that dn/dV is a function of λ and the C_λ form a set of parallel curves, and by the isoperimetric inequality, only if the C_λ are circles. In this case F is a circle and its radius is K . Since the difference of the two sides of (13) represents the integral from 0 to ∞ of a not negative quantity, the same result holds for the equality sign in (13).

The inequality (13) holds for any bounded closed set F . In fact, both sides of the inequality are obtained by the same process in terms of a nested sequence of approximate domains. But other sets besides the circle, if F is not a single continuum, satisfy the relation $\pi K^2 = F$.

3. Applications of conformal mapping. Except for a rotation and translation, a conformal transformation which maps the exterior of a bounded continuum F in the z plane into the exterior of a circle of radius r in the w plane, without distortion at ∞ , may be written in the form

$$(14) \quad z - w = \sum_1^\infty \alpha_n w^{-n}.$$

The radius r is called the Robin constant. We find that $r = K$, where K is the capacity of F . This is seen immediately by comparing (14) with the transformation

$$(15) \quad w = r e^{V(z) + iU(z)}$$

where $U(z) = U(M)$ is the function conjugate to $V(z) = V(M)$, and by noting the value of $|dw/dz|$ in (15) at ∞ .

A direct calculation from (14) in this case shows that

$$\text{meas. } F = \pi r^2 - \pi \sum_1^\infty n r^{-2n} |\alpha_n|^2,$$

so that in particular meas. $F \leq \pi r^2$, which is a special case of (13), and

$$(16) \quad |\alpha_1| \leq r^2.$$

On the other hand, if A is an interior domain in the Z plane, containing the origin, which is mapped conformally on the interior of a circle $|W| = R$ so that $Z=0$ goes into $W=0$ and there is no distortion at $|Z|=0$, the transformation, except for a possible rotation, may be written in the form

$$(17) \quad Z = W + \sum_2^{\infty} \beta_n W^n.$$

Hence

$$A \geq \pi R^2 + \pi \sum_2^{\infty} n |\beta_n|^2 R^{2n} \geq \pi R^2.$$

By means of the transformations

$$w = \frac{1}{W}, \quad z = \frac{1}{Z}$$

the situation is reduced to a transformation of exterior domains, in which $r=1/R$. But the transformation of these exterior domains is given by

$$\begin{aligned} z &= \left(\frac{1}{w} + \sum_2^{\infty} \beta_n w^{-n} \right)^{-1} = w(1 + \beta_2 w^{-1} + \beta_3 w^{-2} + \dots)^{-1} \\ &= w(1 - \beta_2 w^{-1} + \dots). \end{aligned}$$

Consideration of the resulting univalent relation between branches of $z'=z^{1/2}$ and $w'=w^{1/2}$ yields immediately, by (16), the relation

$$|\beta_2| \leq \frac{2}{R}.$$

The above results, which are involved in a direct comparison of the coefficients of the power series are of course well known. But also other known results are obtained in terms of them by means of elementary statements about capacity. From the definition of capacity and the fact that V_0 is the potential due to a distribution of positive mass, there follows easily the statement that if F_1 is contained in F_2 , or is the orthogonal projection of F_2 on a line, then $K(F_1) \leq K(F_2)$. By means of the conformal transformation²

² This is the base transformation for the study of aerofoils.

$$(18) \quad w = \frac{1}{2}(z + (z^2 - a^2)^{1/2})$$

in which $w = (1/2)(x + (x^2 - a^2)^{1/2})$ when $z = x$ is real and $x > a$, the region exterior to the segment $(-a, 0)$, $(a, 0)$ is transformed into the exterior of the circle $|w| = a/2$, without distortion when $z = \infty$. Hence the capacity of a line segment of length $2a$ is $a/2$. Moreover the capacity of a segment of a straight line of length l is less than that of a collection of segments of the line of the same total length l .

Let p be the projection, of linear measure l , on an arbitrary straight line, of the closed bounded set F . Then $K(p) \geq l/4$ and $K(p) \leq K(F)$. Hence

$$(19) \quad l \leq 4K(F).$$

The theorem expressed by (19) was proved in slightly less general form by Pólya, by the application of Tchebytschew polynomials in the definition of transfinite diameter [4].

Consider again an interior domain and the transformation (17). Let d be the minimum distance from O to a point of the boundary of A , R the radius of the corresponding circle in the W plane, there being again no distortion at the origins, which correspond. We have immediately the Koebe result: $d \geq R/4$. In fact, there is a point P on the boundary of A such that $OP = d$. Its image P' is a point of F , where F is the complement of the infinite domain obtained from A by the transformation $z = Z^{-1}$, and the capacity $K(F)$ is the radius $r = 1/R$ of the corresponding circle in the w plane, with $w = W^{-1}$. But, denoting OP' by d' , we have from (19)

$$\frac{1}{d} = d' \leq 4K(F) = 4r = \frac{4}{R},$$

from which the Koebe result follows. That $R/4$ is the best result obtainable follows from the consideration of a region in the z plane which is a narrow approximation to a straight segment.

Again, of all the circles C contained in A and containing O , the radii have an upper bound l , and there is a circle of radius l , whose interior lies in A and such that O lies in the circle or on its circumference. If this situation is interpreted in terms of the transformations $z = Z^{-1}$, $w = W^{-1}$ we find a circle C' , of radius l' , circumscribed about F , but not necessarily the smallest one.

The smallest circle C'' which circumscribes F has a radius l'' which is $\leq \eta/3^{1/2}$, where η is the diameter of F . But $\eta \leq 4K(F)$, so that $l'' \leq 4r/3^{1/2}$. The image of C'' in the z plane is a circle C in A and containing O in its interior. Let P, Q be opposite ends of a diameter of C

through O . Then P' , Q' are opposite ends of a diameter of C'' through O , and

$$l \geq \frac{1}{2} (PO + OQ) = \frac{1}{2} \left(\frac{1}{P'O} + \frac{1}{OQ'} \right) \geq \frac{1}{l''},$$

since the minimum value of $(1/P'O) + (1/OQ')$ is obtained when $P'O = OQ'$. Accordingly

$$l \geq \frac{3^{1/2}}{4r} = \frac{3^{1/2}}{4} R.$$

The value $.43R$ is not, however, the best value which can be obtained for the lower bound of l . By means of stricter inequalities on the coefficients of the Taylor series R . M. Robinson obtains the value $\frac{1}{2}R$. And if the requirement that O shall be contained in the circle is dropped, the radius of the largest circle which lies in A is at least as great as a number k , which as Landau shows [7] satisfies the inequality $k \geq .55R$, and as Robinson shows [8] satisfies also the inequality $k \leq .66R$.

4. Open curves of minimum capacity. The statement (19) about the capacity of the projection of F may be reworded as follows:

Of all continua which contain two fixed points A , B , the straight segment AB has the minimum capacity.

This theorem involves a new orientation towards the question, for it suggests the discussion of the continuum of minimum capacity which contains a given arbitrary collection of fixed points, finite in number. Pólya has considered this problem [4] and obtained the solution of it in the case of $2n$ points spaced regularly, n on each of two concentric circumferences, the desired continuum consisting in this case of the radial lines drawn from the points to the center of the circle.

The situation for n arbitrary points is discussed by Grötsch [9, b]. The aspects of the problem which are involved are seen in the case of three points in the z plane. Consider a circumference of radius r in the w plane, and on it three points $w_1 = r + 0i$, $w_2 = re^{(\phi_1 + \phi_2)i}$, $w_3 = re^{-(\phi_1 + \phi_2)i}$ with ϕ_1, ϕ_2, ϕ_3 positive angles of sum π . A conformal transformation, with unit magnification at ∞ , of the exterior of the unit circle into the exterior of some continuum C in the z plane exists, in which pairs of symmetrical points on the arcs $[0, \phi_1]$, $[0, -\phi_1]$, respectively, adjacent to w_1 , go into single points of the Z plane; and similarly for the arcs adjacent to w_2 and w_3 . The three points $re^{i\phi_1}$, $re^{-i\phi_1}$, $re^{i(\phi_1 + 2\phi_2)}$ thus go into one point z' , and the points w_1, w_2, w_3

correspond to certain distinct non-collinear points z_1, z_2, z_3 . In fact, collinear points z_1, z_2, z_3 would arise only if one of the arcs ϕ_1, ϕ_2, ϕ_3 were zero, as is suggested by (18). The continuum C thus consists of three analytic rays from z' to z_1, z_2, z_3 .

It is proved that C has the least capacity of any continuum which contains the three points z_1, z_2, z_3 regarded as fixed, by reducing the question to the consideration of the maximum sum of "modular ratios" for an area $[9, a]$. This number is in fact equal to the upper bound of the Dirichlet integral of a function u summed over strips into which an area is subdivided, where at the ends of the strips u has the values 0 and 1 respectively, and along the sides of the strips $du/dn=0$, and u satisfies Laplace's equation in each strip. It is then shown by examination of the parameters involved in the transformation, with reference to a principle of continuity in the variation of those parameters, that the above treatment is equivalent to that of the general case. The situation with n points is treated in the same manner, except that intermediate arcs, not abutting on any w_i , arise from the restrictive nature of the conformal transformation.³

5. Open surfaces of minimum capacity. In three dimensions, the simplest problem analogous to those just treated would be to prove the existence of a surface sheet, bounded by a given closed curve in space, which among all such surfaces would have minimum capacity $[10, a, b]$. This problem would be the counterpart of the simplest plane problem, namely, the continuum of minimum capacity containing two given points. But it may be of interest for the record that the author put to himself and solved the space problem first, before becoming familiar with the problem in the plane, and the latter has been introduced in this address, with its fundamental relation to conformal representation, mostly as an extension of the space problem rather than the reverse. In fact, it will be seen later that the statements of the two problems do not exactly correspond.

We consider a simple closed space curve s , itself of zero capacity, in our three-dimensional euclidean space B and topologically equivalent to a circle in the sense that there exists a one-one continuous transformation ξ of a sufficiently large sphere containing s into itself, in such a way that s goes into a circle σ interior to the sphere. For convenience, we suppose that the points of the surface of the sphere are unchanged by the transformation, and we may regard ξ as extended outside the sphere as the identical transformation.

³ The author apologizes for the skimpy treatment of Grötsch's important contribution. Lack of space requires him to refer the reader to the cited memoirs.

Let now S be a piece of surface consisting of a closed set of points which contains s and which, except in the neighborhood of s , consists of a finite number of regular surface elements, each surface element having bounded curvatures. Then, without specifying further for the moment the notion of s being the boundary of S , the analysis of §1 applies, and at every point in a surface element, not on an edge of the element or on s , the conductor potential of S will satisfy the relation (8'). In order, however, not to have to distinguish between local positive and negative normal directions at points of S we shall consider normal directions n_1, n_2 as positively directed away from S and write (8') and (8'') in the form

$$(20) \quad \frac{dV}{dn_1} = \frac{dV}{dn_2},$$

$$(20') \quad \frac{dV}{dn_1} + \frac{dV}{dn_2} = -4\pi\sigma.$$

The function $V(M)$ has a constant value, say 1, on each surface element, and its derivatives are continuous as we approach an interior point of such an element [9, a, p. 165]. If then we define $v_1(M)$ locally as equal to V on the side 1 of the surface element and as equal to $2 - V$ on the side 2 of that element, the function $v_1(M)$ and its first partial derivatives will be continuous across the element, on account of (20); and $v_1(M)$ will be harmonic at points of the element. Consequently the surface element itself will be a piece of analytic surface, being a level surface, without singularities, of a harmonic function. Similarly an extension of V from the side 2 to the side 1 yields a harmonic function v_2 .

It may be shown that there exists a two-valued harmonic function $v(M)$ which has the following properties:

(a) Either branch of the function is harmonically extensible along any curve in space which does not meet s .

(b) The function is bounded and one of its two values tends to 0 at ∞ , the other to 2.

(c) If the two values $v_1(M), v_2(M)$ of v are distinct, a closed path which starts from M and loops s once, returning to M , carries $v_1(M)$ into $v_2(M)$, and vice versa, whereas if the closed path does not loop s it carries each value back into itself.

(l) At every point M not on s , $v_1(M) + v_2(M) = 2$.

(m) Every point of s is a limit point of the locus $v(M) = 1$.

We obtain the function $v(M)$ by applying the Schwarz alternating process (in a form which is equivalent to the sweeping out process,

without its distributions of mass) to a three-dimensional Riemann multiple space $\mathfrak{M}$. In our case the curve s serves as a branch curve, analogous to a branch point in the complex plane, and the Riemann manifold may be constructed simply, by introducing the proper cut space. The necessary cut surface Σ may be drawn, for example, as the image in terms of the transformation ξ of a right circular cylindrical surface with base σ (in the xy plane) and extending to ∞ (in the direction of the positive z axis). The function $v(M)$ is to be single valued in each of the cut spaces. The double cut space may again be replaced by a finite number of simple overlapping spaces by replacing the cylindrical surface by truncated conical surfaces within and outside it, and by means of these overlapping regions the connections of the Riemann manifold may be made across the cut surface.

At ∞ in the first sheet of $\mathfrak{M}$ within the half cylinder Σ , v_1 is to be assigned the value 0 and outside it the value 2, whereas in the second sheet v_2 is to be assigned the value 2 at ∞ within the cylinder and the value 0 at ∞ outside it; and the Schwarz process is employed to draw the two functions v_1, v_2 together through the simple regions which overlap the cuts. There are however one or two difficulties, to expound the details of which would extend unduly the length of this paper. In the first place, the curve s is a curve of zero capacity and hence will not act as a carrier for boundary values, in spite of the fact that $v(M)$ is to assume the value 1 on s ; and in the second place, the surface Σ and the pseudo-conical surfaces are merely the topological images of smooth surfaces and are therefore not necessarily free of sets of points which are irregular for the Dirichlet problem.

The first of these difficulties is overcome by replacing σ by a sequence of tori σ_n condensing to σ as $n \rightarrow \infty$. For each of the corresponding surfaces Σ_n in B a Riemann manifold $\mathfrak{M}_n$ is left, when the interior of Σ_n is cut out, and a function $v^{(n)}(M)$ is set up which has the given properties at ∞ and corresponds to the value 1 on the boundary Σ_n . The second difficulty is handled by making use of the fact that the generalized Dirichlet problem is solvable for domains in which points are simply counted and by performing the analysis in terms of functions which are superharmonic on the interior of the whole Riemann manifold. It is proved that the successive alternate functions are superharmonic even at the exceptional points of the cross cuts which define the overlapping regions.⁴

⁴ Incidentally it may be noted that the above process for solving the generalized Dirichlet problem for a domain on a multiple space of a finite number of leaves enables one to arrive at the Green's function for the domain, which has at a given point a singularity of the form $1/r$ on only one of the leaves.

The sequence $\{v^{(n)}\}$ has a limit function $v(M)$ which is two-valued and of course harmonic in any simply counted bounded domain, so that it evidently satisfies (a). It is easily shown to satisfy (b) on account of the uniformity of convergence except in the neighborhood of s , and (c) and (l) similarly, since each function $v^{(n)}(M)$ has those properties. Because of the non-uniform convergence in the neighborhood of s the property (m) is established with a little more difficulty; it is proved by showing that in any neighborhood of a point of s there lies for each n at least one point P_n , distant from s by more than a given positive δ , where $v^{(n)}(P_n) = 1$. The function $v(M)$ is not necessarily continuous at s , for s has not been restricted as to smoothness, but all of its points are limit points of the surface S which is the locus $v(M) = 1$. It is clear that S occludes no points from ∞ , that is, encloses no portion of space; in fact if the contrary were true we should have $v(M) \equiv 1$ in that portion, and therefore throughout space.

The facts just given about $v(M)$ lead to a more geometrical interpretation of the relation of s to S . Any simple closed curve which loops s once (having no points in common with s) has points in common with S . If in the neighborhood of S the curve is analytic, and cuts S at nonsingular points, it has an odd number of points in common with S , counting multiple intersections in the usual way.

We pass now to the converse statement. The two-valued harmonic function which satisfies the conditions (a), (b), (c) is unique.

The proof of this theorem lies essentially in an extension of Kellogg's uniqueness theorem, which may be stated as follows: Let $u(M)$ be harmonic and bounded on a domain Ω on a manifold $\mathfrak{M}$, which consists of a finite number of sheets, tending to 0 at ∞ if Ω is not a bounded domain, and let m be the upper bound of $u(M)$, $m > 0$. Let F be the set of points Q on the boundary of Ω where $\limsup u(M) \geq m - \epsilon$ as M tends to Q on one or another leaf of $\mathfrak{M}$. Then F considered as a set of points in ordinary space is of positive capacity. The details of the proof are not difficult, but must be omitted here on account of lack of space.

As a consequence of this theorem, since s is of zero capacity, the difference of two functions which satisfy (a), (b), (c) will have to have its upper and lower bounds on $\mathfrak{M}$ both equal to zero.

The surface S , thus determined as the level surface $v = 1$ of the function v , furnishes the minimum capacity among all surfaces S' which have the following properties: S' is bounded and every point of s is a limit point of S' ; except in the neighborhood of s , S' consists of a finite number of regular elements sufficiently smooth for an application of Green's theorem; every closed curve which loops s has a

point in common with S' . Thus S' is a cut surface for $\mathfrak{M}$. The proof depends upon a comparison of the energies of the conductor potential for S and of the conductor potential for S' for the same total mass, in terms of Green's theorem—a comparison which is made possible by evaluating the Dirichlet integral for the conductor potential of S first on $\mathfrak{M}$ and then on S' . The proof has already been given in detail [9, b].

6. Open surfaces of a finite number of sheets. The nature of our problem will be more clearly brought out by considering a slightly more general case, for example, the image s_1, s_2 in B of two non-linked circles σ_1, σ_2 . In this case the connections of the Riemann manifold $\mathfrak{M}$ are easily imagined, since the branches of the two-valued function v are to be interchanged by analytic extension along any circuit which links s_1 or s_2 once, and thus unchanged by a circuit which links them both. If we denote by $u^{(1)}, u^{(2)}$ the two functions $v^{(1)} - 1, v^{(2)} - 1$ where $v^{(1)}$ and $v^{(2)}$ are the two-valued harmonic functions belonging to s_1, s_2 separately, the product $u^{(1)}u^{(2)}$ will be single valued on $\mathfrak{M}$; and also the function $u(M)$, defined by the equation

$$v(M) - 1 = u^{(1)}(M)u^{(2)}(M)u(M),$$

will be single valued in ordinary space. By expressing the fact that the Laplacian of the left-hand member is zero, and utilizing the fact that $1 - u(M)$ vanishes canonically as M tends to ∞ , and is represented by a potential, an integral equation may be found for the unknown function $u(M)$.

But it may also be remarked that if the curves s_1, s_2 are joined by a strip, arbitrarily narrow, so that s is made a simple space curve in the manner of §5, the capacity in the new situation becomes a limiting case of that of the old, since the capacity of the strip may be made arbitrarily small. In other words, the problem of finding a single continuum, or even a single surface, in three dimensions which contains both s_1 and s_2 is illusory; for the capacity of a joining curve may be made zero. Hence our problem refers essentially to the determination of surfaces, finite in number, which will be cut by any circuit which loops a component part of s . In two dimensions there is, however, also a second problem analogous to our three-dimensional one.

We close by discussing qualitatively some simple cases. If the curve s lies in a plane, the surface S is the plane surface enclosed. Nevertheless the surface S in general is evidently not a minimal surface. In fact, if s consists of two circles s_1, s_2 which are parallel cross sections of a long circular cylinder, at a considerable distance from each other,

the minimal surface would consist of the two parallel circular areas. On the other hand, it is evident that the equation (8) is not thus satisfied, so that the two flat discs do not constitute the arrangement for minimum capacity.

In the case of the Plateau problem, if the two circles just described are brought closer together, the discs remain a solution until there is actual contact; on the other hand, if the circles are withdrawn while they are bridged by the constricted longitudinal surface (the catenary of revolution) which satisfies the condition of zero mean curvature, this surface maintains itself until a central node is reached by means of the constriction. With the present problem there is only one surface at any stage, the unstable situation being avoided, in the continuous deformation, by the bulges in the disc-shaped surfaces which are required to minimize the energy. These are caused by the action at a distance which is expressed in terms of (8).

It is seen also by means of (8) that the surface S lies within the closed convex cover of s .

If the two circles of the previous paragraphs are connected along elements of the cylinder by means of a narrow strip, so that s becomes a simple curve, and the strip is twisted so that one of the circles is overturned, we have another figure which changes its character markedly as the circles are brought together. In fact, the surface changes continuously from a two-sided one when the circles are far apart, into a nodal surface and then finally into what is evidently a one-sided one when the circles are sufficiently close so that the curve s can be the boundary of a narrow Möbius strip.

The author must admit, in conclusion, that the methods described in this three-dimensional discussion are quite specialized, and it is hard to see how they can be used in a more general class of problems in the calculus of variations. Perhaps the introduction of a vector potential would offer some advantage for the general point of view.

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CONDUCTION OF HEAT IN REGIONS BOUNDED BY PLANES AND CYLINDERS

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1. **Introduction.** The Green's functions for regions bounded by surfaces of the cylindrical coordinate system are well known.¹ From them solutions may be obtained for problems in which the initial temperature is zero and the surfaces are kept at temperatures which are known functions of the coordinates; the application of Green's function in regions extending to infinity has not been completely studied and the conditions to be satisfied by functions prescribed in such regions are not known. An alternative method of solving such problems consists of using the Laplace transformation and solving the resulting subsidiary equation by separation of variables; here it is found necessary to assume that temperatures given on surfaces extending to infinity satisfy very narrow conditions, such as those of Fourier's or Weber's integral theorems. This is illustrated in §3. Problems of conduction of heat in regions bounded by cylinders and planes, some of which extend to infinity, and with constant surface (and initial) temperatures, or with a radiation boundary condition

$$\frac{\partial v}{\partial n} + h(v - v_0) = 0$$

at a surface, are of considerable importance and the constant surface temperatures do not satisfy the conditions referred to above [cf. §3]. The method given below gives a simple solution of all such problems; the results given form a complete set from which the solutions of all temperature problems in solids bounded by a cylinder and planes perpendicular to its axis, with constant surface (and initial) temperatures, can be written down. Problems involving a radiation boundary condition at some of the surfaces, and problems on the hollow cylinder, may be solved in the same way.

The method was suggested by that given in a previous note² which consisted of the use of a double Laplace transformation; this gives a solution under very wide conditions on the surface temperatures, roughly that they be of exponential type in the space variable. This method also allows verification that the solution obtained does satisfy

¹ Carslaw, *Conduction of Heat* (2d edition, 1921) chaps. 9 and 10; for a discussion using the Laplace transformation see Carslaw and Jaeger, *Journal of the London Mathematical Society*, vol. 15 (1940), p. 273.

² This Bulletin, vol. 46 (1940), pp. 687-693. This paper will be referred to as I.

the differential equation and boundary conditions. In I it was applied to problems on the semi-infinite strip but it may also be applied to problems involving cylindrical boundaries. The form of the solutions obtained by this process suggests a simple method of solving problems of the types referred to above. This is here applied to obtain solutions for regions bounded by surfaces of the cylindrical coordinate system with constant surface temperatures. In §2 one such problem is solved in detail to illustrate the method, and in §3 the solution is compared with that obtained by direct separation of variables. Other problems are solved briefly in §§4-7.

The method given here is formal both in the application of the Laplace transformation and in that infinite processes are interchanged without justification; in special cases it can be verified directly that the results do satisfy the differential equations and initial and boundary conditions; as remarked above this is best done by the method of I, when applicable. The use of the Laplace transformation and the method of evaluating and justifying solutions by the use of the inversion theorem followed by contour integration will be assumed known³ and used freely without comment.

2. Heat conduction in the semi-infinite cylinder $z > 0$, $0 \leq r < a$; $r = a$ maintained at unit temperature for $z > 0$ and $t > 0$; $z = 0$ maintained at zero for $0 \leq r < a$, $t > 0$; the initial temperature zero. We have to solve

$$(1) \quad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{1}{\kappa} \frac{\partial v}{\partial t} = 0, \quad 0 \leq r < a, z > 0, t > 0,$$

with

$$(2) \quad v = 1, \quad r = a, \quad z > 0, t > 0;$$

$$(3) \quad v = 0, \quad z = 0, \quad 0 \leq r < a, t > 0;$$

$$(4) \quad v = 0, \quad t = 0, \quad z > 0, 0 \leq r < a.$$

We multiply (1) and its boundary conditions (2) and (3) by e^{-pt} , $R(p) > 0$, then writing

$$(5) \quad \bar{v}(p) = \int_0^\infty e^{-pt} v(t) dt$$

for the Laplace transform of v we obtain the subsidiary equation

³ Cf. Carslaw and Jaeger, Proceedings of the Cambridge Philosophical Society, vol. 35 (1939), p. 394, and this Bulletin, vol. 45 (1939), p. 407.

$$(6) \quad \frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2 \bar{v} = 0, \quad 0 \leq r < a, z > 0,$$

where

$$(7) \quad q^2 = p/\kappa.$$

This notation will be used throughout. Equation (6) is to be solved with boundary conditions

$$(8) \quad \bar{v} = 1/p, \quad r = a, \quad z > 0;$$

$$(9) \quad \bar{v} = 0, \quad z = 0, \quad 0 \leq r < a.$$

Equation (6) is satisfied by terms of type

$$(10) \quad J_0(r\alpha_m) e^{-s(q^2 + \alpha_m^2)^{1/2}}$$

where α_m , $m = 1, 2, \dots$, are the positive roots of

$$(11) \quad J_0(a\alpha) = 0.$$

As suggested by the form of the results obtained in I, we seek a solution of (6) of type

$$(12) \quad \frac{1}{p} \frac{I_0(qr)}{I_0(qa)} + \sum_{m=1}^{\infty} A_m J_0(r\alpha_m) e^{-s(q^2 + \alpha_m^2)^{1/2}}$$

in which the first term is the solution of the subsidiary equation for the infinite cylinder with unit surface temperature and the second a series of correcting terms which satisfy (6) and vanish when $r = a$, and whose coefficients A_m are to be chosen so that (12) satisfies (9). This requires

$$(13) \quad \frac{1}{p} \frac{I_0(qr)}{I_0(qa)} + \sum_{m=1}^{\infty} A_m J_0(r\alpha_m) = 0, \quad 0 \leq r < a,$$

that is,

$$A_m = - \frac{2\alpha_m}{ap(q^2 + \alpha_m^2)J_1(a\alpha_m)}.$$

Thus

$$(14) \quad \bar{v} = \frac{I_0(qr)}{pI_0(qa)} - \frac{2}{a} \sum_{m=1}^{\infty} \frac{\alpha_m}{p(q^2 + \alpha_m^2)} \frac{J_0(r\alpha_m)}{J_1(a\alpha_m)} e^{-s(\alpha_m^2 + q^2)^{1/2}}$$

satisfies the subsidiary equation (6) and its boundary conditions (8) and (9). As remarked in §1 this result is obtained directly by the method of I.

From (14) $v(t)$ is found in the usual way⁴ by the use of the inversion theorem for the Laplace transformation and we have finally

$$(15) \quad v(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \bar{v}(\lambda) d\lambda = 1 - \frac{2}{a} \sum_{m=1}^{\infty} \frac{J_0(r\alpha_m) e^{-s\alpha_m}}{\alpha_m J_1(a\alpha_m)} - \frac{4}{\pi a} \sum_{m=1}^{\infty} \frac{\alpha_m J_0(r\alpha_m)}{J_1(a\alpha_m)} \int_0^{\infty} \frac{e^{-s t(u^2 + \alpha_m^2)} \sin zu \, du}{u(u^2 + \alpha_m^2)}.$$

This may easily be put in the more suitable form for computation

$$(16) \quad v = 1 - \frac{1}{a} \sum_{m=1}^{\infty} \frac{J_0(r\alpha_m)}{\alpha_m J_1(a\alpha_m)} \left\{ 2e^{-s\alpha_m^2 t} \operatorname{erf} \frac{z}{2(\kappa t)^{1/2}} + e^{s\alpha_m} \operatorname{erfc} \left[\frac{z}{2(\kappa t)^{1/2}} + \alpha_m(\kappa t)^{1/2} \right] + e^{-s\alpha_m} \operatorname{erfc} \left[\frac{z}{2(\kappa t)^{1/2}} - \alpha_m(\kappa t)^{1/2} \right] \right\}.$$

3. **Comparison of the solution of §2 with that obtained formally by direct separation of variables in the subsidiary equation.** Consider the problem of §2 but with $r=a$ maintained at $f(z)$ in place of unity for $t>0$, $z>0$. The subsidiary equation (6) has to be solved with boundary condition (9) and with

$$(17) \quad \bar{v} = \frac{1}{p} f(z), \quad r = a, z > 0.$$

We seek a solution composed of terms of type $I_0(q'r) \sin uz$, where

$$(18) \quad q' = (q^2 + u^2)^{1/2}.$$

Such a solution will be

$$\bar{v} = \int_0^{\infty} g(u) I_0(q'r) \sin uz \, du.$$

This satisfies (6) and (9), and (17) requires

$$\int_0^{\infty} g(u) I_0(q'a) \sin uz \, du = \frac{1}{p} f(z).$$

If we assume that $f(z)$ satisfies conditions for Fourier's integral theorem, for example, $f(z)$ satisfies Dirichlet's conditions and $\int_0^{\infty} f(z) dz$ converges absolutely, $g(u)$ can be determined and we obtain

⁴ Cf. Carslaw and Jaeger, loc. cit.

$$(19) \quad \bar{v} = \frac{2}{\pi} \int_0^\infty \frac{I_0(rq')}{pI_0(aq')} \sin uz \, du \int_0^\infty f(z') \sin uz' \, dz',$$

and hence, using the inversion theorem for the Laplace transformation,

$$(20) \quad v = \frac{2}{\pi} \int_0^\infty \left\{ \frac{I_0(ru)}{I_0(au)} - \frac{2}{a} \sum_{m=1}^\infty \frac{\alpha_m e^{-z(\alpha_m^2 + u^2)} J_0(r\alpha_m)}{(\alpha_m^2 + u^2) J_1(a\alpha_m)} \right\} \sin uz \, du \int_0^\infty f(z') \sin uz' \, dz',$$

where the α_m , $m=1, 2, \dots$, are the positive roots of $J_0(a\alpha)=0$ and the method of solution is available only for the above restricted values of $f(z)$. However, if we invert the orders of integration in (19) convergent solutions may be obtained for functions $f(z)$ which do not satisfy the conditions above, for example (15) may be obtained in this way.

4. The region $r > a$, $0 < z < l$; $z=l$ kept at unity for $t > 0$; the other surfaces at zero; the initial temperature zero. Here the subsidiary equation is

$$(21) \quad \frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2 \bar{v} = 0, \quad r > a, \quad 0 < z < l,$$

to be solved with

$$(22) \quad \bar{v} = 1/p, \quad z = l, \quad r > a;$$

$$(23) \quad \bar{v} = 0, \quad z = 0, \quad r > a;$$

$$(24) \quad \bar{v} = 0, \quad r = a, \quad 0 < z < l.$$

The solution of the subsidiary equation for the infinite flat plate $0 < z < l$ with boundary conditions (22) and (23) is $(\sinh qz)/(p \sinh ql)$, so we choose for the solution of (21)

$$(25) \quad \bar{v} = \frac{\sinh qz}{p \sinh ql} + \sum_{n=1}^\infty A_n \sin \frac{n\pi z}{l} K_0(rq_n)$$

where

$$(26) \quad q_n^2 = (q^2 + n^2\pi^2/l^2)^{1/2}.$$

This satisfies (21), (22), and (23), and (24) requires

$$\frac{\sinh qz}{p \sinh ql} + \sum_{n=1}^\infty A_n \sin \frac{n\pi z}{l} K_0(aq_n) = 0$$

that is,

$$A_n = \frac{2n\pi(-)^n}{pl^2q_n^2K_0(aq_n)}.$$

Thus

$$(27) \quad \bar{v} = \frac{\sinh qz}{p \sinh ql} + \frac{2\pi}{pl^2} \sum_{n=1}^{\infty} \frac{n(-)^n K_0(rq_n)}{q_n^2 K_0(aq_n)} \sin \frac{n\pi z}{l}.$$

And, using the inversion theorem for the Laplace transformation,

$$(28) \quad v = \frac{z}{l} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-)^n K_0(n\pi r/l)}{n K_0(n\pi a/l)} \sin \frac{n\pi z}{l} \\ - \frac{4}{l^2} \sum_{n=1}^{\infty} \left\{ n(-)^n e^{-\kappa n^2 x^2/l^2} \sin \frac{n\pi z}{l} \right. \\ \left. \int_0^{\infty} \frac{e^{-\kappa u^2} C_0(ur, ua) du}{u(u^2 + n^2\pi^2/l^2) [J_0^2(ua) + Y_0^2(ua)]} \right\},$$

where

$$(29) \quad C_0(x, y) = J_0(x)Y_0(y) - Y_0(x)J_0(y).$$

5. The region $r > a, z > 0; z=0$ kept at unity for $t > 0, r > a; r=a$ kept at zero for $z > 0, t > 0$; the initial temperature zero. Here

$$(30) \quad \bar{v} = \frac{1}{p} e^{-qz} - \frac{2}{\pi p} \int_0^{\infty} \frac{u \sin uz}{u^2 + q^2} \frac{K_0(rq')}{K_0(aq')} du,$$

where q and q' are defined in (7) and (18) respectively, satisfies the subsidiary equation and its boundary condition at $z=0$. It also satisfies the boundary condition at $r=a$ since

$$\frac{2}{\pi} \int_0^{\infty} \frac{u \sin uz}{u^2 + q^2} du = e^{-qz}.$$

Using the inversion theorem we obtain from (30)

$$(31) \quad v = 1 - \frac{2}{\pi} \int_0^{\infty} \frac{K_0(ur) \sin uz}{u K_0(ua)} du + \frac{4}{\pi^2} \int_0^{\infty} e^{-\kappa u^2} u \sin uz du \\ \int_0^{\infty} \frac{e^{-\kappa u'^2} C_0(ru', au') du'}{u'(u'^2 + u^2) [J_0^2(au') + Y_0^2(au')]}.$$

6. The region $r > a, z > 0; r=a$ kept at unity for $z > 0, t > 0; z=0$ kept at zero for $r > a, t > 0$; the initial temperature zero. Here



$$(32) \quad \bar{v} = \frac{1}{p} \frac{K_0(qr)}{K_0(qa)} + \frac{2}{\pi p} \int_0^\infty e^{-sq'} \frac{C_0(ur, ua) u du}{(q^2 + u^2) [J_0^2(ua) + Y_0^2(ua)]}$$

satisfies the subsidiary equation and its boundary condition at $r=a$. It also satisfies the boundary condition at $z=0$, since, by Weber's integral theorem,

$$\frac{K_0(qr)}{K_0(qa)} + \frac{2}{\pi} \int_0^\infty \frac{C_0(ur, ua) u du}{(q^2 + u^2) [J_0^2(ua) + Y_0^2(ua)]} = 0.$$

It follows that

$$(33) \quad \begin{aligned} v = 1 + & \frac{2}{\pi} \int_0^\infty \frac{e^{-uz} C_0(ru, au) du}{u [J_0^2(au) + Y_0^2(au)]} \\ & + \frac{4}{\pi^2} \int_0^\infty \frac{e^{-\epsilon u^2} C_0(ru, au) u du}{J_0^2(au) + Y_0^2(au)} \int_a^\infty \frac{e^{-\epsilon u'^2} \sin u'z du'}{u' (u^2 + u'^2)}. \end{aligned}$$

7. The finite cylinder and similar problems. In this section four results are given for completeness which are obtained by the method of §2; these problems may also be solved by the method of §3, and in all cases the transient parts of the solutions are identical, but the steady state parts are obtained in a different form.

(i) *The region $0 \leq r < a$, $z > 0$; $z=0$ kept at unity for $t > 0$, $0 \leq r < a$; $r=a$ kept at zero for $z > 0$, $t > 0$; the initial temperature zero:*

$$(34) \quad \bar{v} = \frac{1}{p} e^{-qz} - \frac{2}{\pi p} \int_0^\infty \frac{u \sin uz}{q^2 + u^2} \frac{I_0(rq')}{I_0(aq')} du,$$

where q and q' are defined in (7) and (18);

$$(35) \quad \begin{aligned} v = 1 - & \frac{2}{\pi} \int_0^\infty \frac{I_0(ur) \sin uz du}{u I_0(ua)} \\ & - \frac{4}{\pi a} \sum_{m=1}^\infty \frac{e^{-\epsilon \alpha_m^2} J_0(r \alpha_m)}{\alpha_m J_1(a \alpha_m)} \int_0^\infty \frac{e^{-\epsilon u^2} u \sin uz du}{(\alpha_m^2 + u^2)}, \end{aligned}$$

where the α_m are the positive roots of $J_0(a\alpha) = 0$.

(ii) *The region $r > a$, $0 < z < l$; $r=a$ kept at unity for $t > 0$, $0 < z < l$; the other surfaces kept at zero for $t > 0$; the initial temperature zero:*

$$(36) \quad \bar{v} = \frac{K_0(qr)}{p K_0(qa)} + \int_0^\infty \frac{2 \cosh q'(\frac{1}{2}l - z) C_0(ru, au) u du}{\pi p q'^2 \cosh \frac{1}{2} q' l [J_0^2(ua) + Y_0^2(ua)]},$$

$$\begin{aligned}
 (37) \quad v = 1 + \frac{2}{\pi} \int_0^\infty \frac{\cosh u(\frac{1}{2}l - z) C_0(ru, au) du}{u \cosh \frac{1}{2}lu [J_0^2(au) + Y_0^2(au)]} \\
 + \frac{8}{\pi^2} \sum_{n=0}^\infty \left\{ e^{-\kappa(2n+1)^2\pi^2 t/l^2} \frac{\sin(2n+1)\pi z/l}{(2n+1)} \right. \\
 \left. \int_0^\infty \frac{e^{-\kappa u^2 t} C_0(ru, au) u du}{[u^2 + (2n+1)^2\pi^2/l^2] [J_0^2(au) + Y_0^2(au)]} \right\}.
 \end{aligned}$$

(iii) The finite cylinder $0 \leq r < a$, $0 < z < l$; $r=a$ kept at unity and the other surfaces at zero for $t > 0$; the initial temperature zero:

$$(38) \quad \bar{v} = \frac{I_0(qr)}{pI_0(qa)} - \frac{2}{a} \sum_{m=1}^\infty \frac{\alpha_m J_0(r\alpha_m)}{p q_m^2 J_1(a\alpha_m)} \frac{\cosh(\frac{1}{2}l - z)q_m}{\cosh \frac{1}{2}l q_m},$$

where α_m , $m=1, 2, \dots$, are the positive roots of $J_0(a\alpha)=0$ and $q_m = (q^2 + \alpha_m^2)^{1/2}$;

$$\begin{aligned}
 (39) \quad v = 1 - \frac{2}{a} \sum_{m=1}^\infty \frac{J_0(r\alpha_m)}{\alpha_m J_1(a\alpha_m)} \frac{\cosh(\frac{1}{2}l - z)\alpha_m}{\cosh \frac{1}{2}l \alpha_m} \\
 - \frac{8}{a\pi} \sum_{m=1}^\infty \left\{ \frac{\alpha_m J_0(r\alpha_m)}{J_1(a\alpha_m)} e^{-\kappa \alpha_m^2 t} \right. \\
 \left. \sum_{n=0}^\infty \frac{e^{-\kappa(2n+1)^2\pi^2 t/l^2} \sin(2n+1)\pi z/l}{(2n+1)[\alpha_m^2 + (2n+1)^2\pi^2/l^2]} \right\}.
 \end{aligned}$$

(iv) The finite cylinder $0 \leq r < a$, $0 < z < l$; $z=l$ kept at unity and the other surfaces at zero for $t > 0$; the initial temperature zero:

$$(40) \quad \bar{v} = \frac{\sinh qz}{p \sinh ql} + \frac{2\pi}{pl^2} \sum_{n=1}^\infty \frac{n(-)^n I_0(rq_n)}{q_n^2 I_0(aq_n)} \sin \frac{n\pi z}{l},$$

where $q_n = (q^2 + n^2\pi^2/l^2)^{1/2}$;

$$\begin{aligned}
 (41) \quad v = \frac{z}{l} + \frac{2}{\pi} \sum_{n=1}^\infty \frac{(-)^n I_0(n\pi r/l)}{n I_0(n\pi a/l)} \sin \frac{n\pi z}{l} \\
 + \frac{4\pi}{a} \sum_{n=1}^\infty n(-)^n \sin \frac{n\pi z}{l} \sum_{m=1}^\infty \frac{J_0(r\alpha_m) e^{-\kappa t(\alpha_m^2 + n^2\pi^2/l^2)}}{\alpha_m(l^2\alpha_m^2 + n^2\pi^2) J_1(a\alpha_m)},
 \end{aligned}$$

where α_m , $m=1, 2, \dots$, are the positive roots of $J_0(a\alpha)=0$.

ON A THEOREM BY J. L. WALSH CONCERNING THE MODULI OF ROOTS OF ALGEBRAIC EQUATIONS

ALEXANDER OSTROWSKI

In 1881 A. E. Pellet published¹ the following very useful theorem:

If the polynomial

$$(1) \quad F(z) \equiv |a_0| + |a_1|z + |a_2|z^2 + \cdots + |a_{k-1}|z^{k-1} \\ - |a_k|z^k + |a_{k+1}|z^{k+1} + \cdots + |a_n|z^n, \\ 0 < k < n, a_0 a_n \neq 0,$$

has two positive roots x_1 and x_2 ($x_1 < x_2$), then the polynomial

$$(2) \quad f(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

has no roots in the annulus $x_1 < |z| < x_2$ and precisely k roots in the circle $|z| \leq x_1$.

While Pellet's proof for his theorem utilizes the theorem of Rouché, J. L. Walsh published in 1924² another more direct proof and established in the same memoir a sort of converse of Pellet's theorem. To formulate his result, consider the set $\mathfrak{F}$ of all polynomials

$$(3) \quad f(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

which correspond to given moduli of coefficients. All polynomials of $\mathfrak{F}$ are obtained from one of them, $f(z)$, if the factors $\epsilon_0, \epsilon_1, \dots, \epsilon_n$ in the expression

$$(4) \quad \epsilon_0 a_0 + \epsilon_1 a_1 z + \epsilon_2 a_2 z^2 + \cdots + \epsilon_n a_n z^n$$

assume independently all values of modulus 1. Let $\mathfrak{M}$ be the set of roots of all polynomials in $\mathfrak{F}$. It is immediately seen that if $\mathfrak{M}$ contains a number α it also contains all numbers with the modulus $|\alpha|$.

It was proved by Cauchy that all roots of (4) lie on or within the circle $|z| = y_1$, where y_1 is the single positive root of the polynomial

$$|a_0| + |a_1|z + \cdots + |a_{n-1}|z^{n-1} - |a_n|z^n$$

and that all roots of (4) lie on or are exterior to the circle $|z| = y_2$, where y_2 is the single positive root of the polynomial

¹ A. E. Pellet: *Sur un mode de séparation des racines des équations et la formule de Lagrange*, Bulletin des Sciences Mathématiques, (2), vol. 5 (1881), pp. 393-395.

² J. L. Walsh: *On Pellet's theorem concerning the roots of a polynomial*, Annals of Mathematics, vol. 26 (1924), pp. 59-64.

$$|a_0| - |a_1|z - \cdots - |a_n|z^n.$$

Since the numbers y_1 and y_2 obviously belong to $\mathfrak{M}$, it follows that $\mathfrak{M}$ lies in the closed annulus $y_2 \leq |z| \leq y_1$.

Suppose now that $a > 0$ is not contained in $\mathfrak{M}$. Then, if C is the circle $|z| = a$, all polynomials of $\mathfrak{F}$ contain the same number k of roots within C . Indeed, if we vary continuously the factors ϵ_r in the representation (4), the roots of (4) vary also continuously and their number within C remains the same since none of them is able to pass C .

Now the theorem of Walsh runs as follows:

Any positive a that is not contained in $\mathfrak{M}$, and for which the number of roots of (4) within the circle $|z| = a$ is precisely k ($0 < k < n$), is contained between the two positive roots of the polynomial (1), that is to say:

$$(5) \quad |a_k| a^k > (|a_0| + \cdots + |a_{k-1}| a^{k-1}) \\ + (|a_{k+1}| a^{k+1} + \cdots + |a_n| a^n).$$

As the proof given by Walsh of this important result is not complete,³ we give in what follows another proof proceeding on different lines.

Suppose that, contrary to (5),

$$(6) \quad |a_k| a^k \leq \sum_{r=0, r \neq k}^n |a_r| a^r;$$

then, as a is not contained in $\mathfrak{M}$, we have even

$$(7) \quad |a_k| a^k < \sum_{r=0, r \neq k}^n |a_r| a^r.$$

On the other hand it follows from our hypothesis that if ϵ_r run independently through all constants of modulus 1,

$$(8) \quad \sum_{r=0}^n \epsilon_r a_r a^r \neq 0,$$

and therefore

³ Walsh allows in his proof the roots of $f(z)$, which are in absolute value less than a , to vary continuously and monotonically (in absolute value) and to approach 0. But during this variation the polynomial $f(z)$ does not necessarily remain in the set $\mathfrak{F}$ and the corresponding sets $\mathfrak{F}_0$ for the polynomials thus obtained could very well contain a in the set of the roots, so that the expression (1) need not remain positive for $z = a$, as is assumed in Walsh's proof.

$$(9) \quad |a_k| a^k \neq \left| \sum_{p=0, p \neq k}^n \epsilon_p a_p a^p \right|.$$

But from (7) and (9) it follows that for all values of ϵ_p in question

$$(10) \quad |a_k| a^k < \left| \sum_{p=0, p \neq k}^n \epsilon_p a_p a^p \right|.$$

For, if we have for a particular set of ϵ_p -values, ϵ_p^0 :

$$(11) \quad |a_k| a^k > \left| \sum_{p=0, p \neq k}^n \epsilon_p^0 a_p a^p \right|,$$

we see from (7) and (11), that the right-hand side of (10) becomes equal to $|a_k| a^k$ for some other set of ϵ_p , contrary to (9).

Now it follows from (10) that the polynomials

$$(12) \quad f^*(z) = \sum_{p=0, p \neq k}^n \epsilon_p a_p z^p$$

have no roots at a and therefore no roots on the circle C . On the other hand, we have obviously by (10) everywhere on C : $|f^*(z)| > |a_k z^k|$. Hence, by the theorem of Rouché, since $f^*(z) - a_k z^k$ has exactly k roots inside C , the same is true for any polynomial $f^*(z)$.

The result arrived at may be announced in the following form:

If $\epsilon_0, \dots, \epsilon_{k-1}, \epsilon_{k+1}, \dots, \epsilon_n$ run independently through all values of modulus 1, let

$$\phi(z) = \sum_{p=0}^{k-1} \epsilon_p a_p z^p, \quad \psi(z) = \sum_{p=k+1}^n \epsilon_p a_p z^p;$$

then the difference $\phi(z) - \psi(z)$ does not vanish on C and has exactly k roots inside C .

It follows, that in particular

$$\left| \sum_{p=0}^{k-1} \epsilon_p a_p a^p \right| \neq \left| \sum_{p=k+1}^n \epsilon_p a_p a^p \right|.$$

But then it is impossible that we have simultaneously for one particular set ϵ'_p of ϵ_p

$$\left| \sum_{p=0}^{k-1} \epsilon'_p a_p a^p \right| > \left| \sum_{p=k+1}^n \epsilon'_p a_p a^p \right|$$

and for another, ϵ''_p ,

$$\left| \sum_{r=0}^{k-1} \epsilon_r'' a_r a^r \right| < \left| \sum_{r=k+1}^n \epsilon_r'' a_r a^r \right|.$$

Hence only two cases are possible:

A. *We have always*

$$\left| \sum_{r=0}^{k-1} \epsilon_r a_r a^r \right| > \left| \sum_{r=k+1}^n \epsilon_r a_r a^r \right|,$$

and therefore everywhere on C : $|\phi(z)| > |\psi(z)|$.

But then, by the theorem of Rouché, $\phi(z)$ has inside C the same number of roots as $\phi(z) - \psi(z)$, that is k , and this is impossible, $\phi(z)$ being of degree $k-1$.

B. *We have always*

$$\left| \sum_{r=0}^{k-1} \epsilon_r a_r a^r \right| < \left| \sum_{r=k+1}^n \epsilon_r a_r a^r \right|,$$

and therefore everywhere on C : $|\phi(z)| < |\psi(z)|$.

But then $\psi(z)$ would have inside C exactly k roots, while $\psi(z)$ vanishes at $z=0$ with the multiplicity $k+1$ at least.

We see that (7) and (6) are impossible and the proof of (5) is completed.

The theorems of Pellet and Walsh dealt with in the preceding paragraphs allow us to describe immediately the set $\mathfrak{M}$.

Consider the $n+1$ equations:

$$(13) \quad |a_n| z^n - \sum_{r=0}^{n-1} |a_r| z^r = 0,$$

$$(14) \quad \sum_{r=k+1}^n |a_r| z^r - |a_k| z^k + \sum_{r=0}^{k-1} |a_r| z^r = 0, \quad k = 1, \dots, n-1,$$

$$(15) \quad \sum_{r=1}^n |a_r| z^r - |a_0| = 0.$$

If, as we will assume, $a_0 a_n \neq 0$, each of the equations (13), (15) possesses one positive root ρ resp. ρ_0 , and we have $\rho_0 \leq \rho$ since ρ is the exact upper and ρ_0 the exact lower limit for the moduli of $\mathfrak{M}$. As to the equations (14), every one of them possesses either two positive roots or exactly one double positive root or no positive roots at all. Strike out the equations (14) corresponding to the two last cases; each of the remaining equations (14) possesses two positive roots, x_k, x_k'

$(x_k < x'_k)$. Then we obtain $\mathfrak{M}$ by removing from the plane of z the two circular domains $|z| < \rho_0$ and $|z| > \rho$ and all annuli $x_k < |z| < x'_k$, corresponding to the equations (14) with two different positive roots.

It is then a consequence of Walsh's theorem that the different (open) intervals (x_k, x'_k) have no points in common and lie in the interval (ρ_0, ρ) ; moreover, if for two of these intervals (x_k, x'_k) , (x_m, x'_m) we have $k < m$, then we have certainly $x'_k \leq x_m$, and it is easily seen that we have even $x'_k < x_m$.

As Walsh remarks, his proof of Pellet's theorem remains valid also in the case of a power-series and of its roots inside the circle of convergence. It is hardly necessary to remark that our proof of Walsh's theorem also applies *mutatis mutandis* to a power series, if we only consider its roots within the circle of convergence.

UNIVERSITY OF BASEL

SOME EXCEPTIONAL VALUES OF THE LIMIT OF THE RATIO OF ARC TO CHORD

RICHARD COHN

It was observed by E. Kasner¹ that in the complex euclidean plane the limiting value of the ratio of the arc of a curve to its chord, while one end point of the arc is fixed, and the other approaches it along the curve, is not always unity; but assumes for analytic curves tangent to a minimal line, a sequence of real values, .94 . . . , .86 . . . , .80, ... These values are functions of the order of contact only, and approach zero as the latter increases. In this note we shall describe two similar situations which occur in real spaces.

The problem in the case of the K plane² has been worked out in Professor Kasner's Seminar in Geometry.³ In this plane the length of the curve $y=f(x)$ passing between points of abscissae x_1, x_2 , in that order, is given by

$$(1) \quad l = \int_{x_1}^{x_2} \left(\frac{dy}{dx} \right)^{-1} dx.$$

¹ E. Kasner, *The ratio of the arc to the chord of an analytic curve need not approach unity*, this Bulletin, vol. 21 (1914), pp. 524-531. Similar questions for three dimensions are discussed in E. Kasner, *Complex geometry and relativity, theory of the "rac" curvature*, Proceedings of the National Academy of Sciences, 1932, p. 267.

² Kasner, *Trihornometry, a new chapter of conformal geometry*, Proceedings of the National Academy of Sciences, vol. 23, p. 346.

³ R. Coleman, S. Jablon and D. Mittleman obtained the results for the K plane given below.

The distance from the point (x_1, y_1) to the point (x_2, y_2) is defined as the length of the straight line passing from the first of these points to the second and is therefore given by the equation

$$(2) \quad c = \frac{(x_2 - x_1)^2}{y_2 - y_1}.$$

We see that lines parallel to the x -axis have infinite length while those parallel to the y -axis are the minimal lines or lines of zero length. We shall now investigate the ratio of the length of an analytic arc to the distance between its end points, both length and distance being used in the sense just defined.

The most interesting cases will be those of curves tangent to a minimal line. We will obviously lose nothing by choosing the origin for the point of contact. We will write the equation of our curve in the form $x = a_n y^n + a_{n+1} y^{n+1} + \dots$, $a_n \neq 0$. Evidently this includes the normal as well as the exceptional situations; for when $n=1$ the curve cuts the minimal line, for greater values of n it is tangent to it.

We may write for the length of the curve from the origin to the point (X, Y) the equation $l = \int_0^Y (dx/dy)^2 dy$. This is obtained easily from the expression (1) for the length. Since $dx/dy = na_n y^{n-1} + \dots$, this becomes

$$(3) \quad l = \int_0^Y (na_n y^{n-1} + \dots)^2 dy = \int_0^Y (n^2 a_n^2 y^{2n-2} + \dots) dy,$$

$$l = \frac{n^2}{2n-1} a_n^2 Y^{2n-1} + \dots.$$

The distance from the origin to the point (X, Y) is seen to be

$$(4) \quad c = \frac{X^2}{Y} = \frac{(a_n Y^n + \dots)^2}{Y} = a_n^2 Y^{2n-1} + \dots.$$

The first terms of these expansions do not vanish, so that as Y approaches zero, l/c approaches their quotient $n^2/(2n-1)$ as limit. *For $n=1$ this is unity, but for curves tangent to the minimal line this limit assumes a sequence of values $4/3, 9/5, 25/4, \dots$, increasing towards infinity with the order of contact.*

There remains the case of curves tangent to a line of infinite length. These must be written in the form $y = a_n x^n + \dots$, $a_n \neq 0$, $n > 1$. A calculation similar to the preceding one shows that such curves have infinite arc length measured from the point of tangency to any other point, while the corresponding chords are finite.

In the affine differential geometry of the plane which has been developed by Blaschke⁴ and others we work with elements consisting of a point and a slope. For a curve to pass from the element consisting of the point (x_1, y_1) and the slope y_1' to the element consisting of the point (x_2, y_2) and the slope y_2' it must pass through each point and have at each the corresponding slope. If $y=f(x)$ is the equation of such a curve, the affine length of the arc between these elements is defined by

$$(5) \quad l = \int_{x_1}^{x_2} \left(\frac{d^2 y}{dx^2} \right)^{1/3} dx.$$

This integral is invariant under an area-preserving affine transformation. With the given end-conditions it becomes stationary when the arc is a parabola. Its value in this case is a relative maximum and is called the affine distance between the elements.

I have carried over the problem to this case by considering the limit of the ratio of the affine length of an arc to the affine distance between the elements tangent to the arc at its end points, as one end point is fixed and the other approaches it along the curve.

It will be observed that all straight lines have zero affine length, and it is to be expected, by analogy with the preceding cases, that unusual values of the limit will occur for those curves and only those which have high order of contact with a straight line. That this is indeed the case is shown in the following calculations.

By means of a suitable translation and rotation, which cannot affect the quantities under investigation, we write any analytic curve in the form $y = a_n x^n + a_{n+1} x^{n+1} + \dots$, $a_n \neq 0$, $n \geq 2$. The affine length of this curve from the element at the origin with slope zero to the element tangent to the curve at the point (X, Y) is found by means of (5) to be given by

$$(6) \quad \begin{aligned} l &= \int_0^X \left(\frac{d^2 y}{dx^2} \right)^{1/3} dx = \int_0^X (n(n-1)a_n x^{n-2} + \dots)^{1/3} dx, \\ l &= \int_0^X ([n(n-1)a_n]^{1/3} x^{(n-2)/3} + \dots) dx, \\ l &= \frac{[n(n-1)a_n]^{1/3}}{(n+1)/3} X^{(n+1)/3} + \dots \end{aligned}$$

⁴ W. Blaschke, *Differentialgeometrie*, vol. 2, chap. 1. (5) and (7) below are easily obtained from the equations given by Blaschke in vector notation.

The general formula for the affine distance between the elements consisting of the points (x_1, y_1) , (x_2, y_2) with the slopes y'_1 and y'_2 respectively is

$$(7) \quad c = \left[\frac{4[(y_2 - y_1) - y'_1(x_2 - x_1)][y'_2(x_2 - x_1) - (y_2 - y_1)]}{y'_2 - y'_1} \right]^{1/3}.$$

In the case under consideration this becomes

$$(8) \quad c = \left[\frac{4Y}{Y'} (XY' - Y) \right]^{1/3}, \quad Y' = \frac{dy}{dx} \Big|_{x=X} = na_n X^{n-1} + \dots,$$

$$c = \left[\frac{(4a_n X^n + \dots)([na_n X^n + \dots] - [a_n X^n + \dots])}{na_n X^{n-1} + \dots} \right]^{1/3},$$

$$c = \left[\frac{4(n-1)a_n}{n} \right]^{1/3} X^{(n+1)/3} + \dots.$$

We observe that the first terms of the expansions for l and c do not vanish. Consequently as X approaches zero the ratio of these quantities approaches

$$\frac{3}{n+1} \left[\frac{n^2}{4} \right]^{1/3}.$$

Thus the limiting value of the ratio of affine arc to affine chord will be unity for the case $n=2$, but for curves with a higher order of contact with a straight line it will assume a sequence of values .98 . . . , .95 . . . , .92 . . . , . . . , dependent only on the order of contact and decreasing towards zero as the latter increases.

We have thus seen that there is a close agreement in the behavior of the limit of the ratio of the arc and chord of an analytic curve in the complex euclidean, the Kasner and the affine plane. *In each case this quantity is unity for most curves, but may also have one of a denumerable set of other values. These values occur for curves with higher than the least possible order of contact with a minimal line, and depend only on that order of contact.*

A NOTE ON FUNCTIONS OF EXPONENTIAL TYPE¹

R. P. BOAS, JR.

An entire function $f(z)$ is said to be of exponential type at most T if

$$(1) \quad \limsup_{n \rightarrow \infty} |f^{(n)}(z)|^{1/n} \leq T$$

for some z (and hence for every z , uniformly for z in any bounded set). An equivalent condition² is that for each positive ϵ

$$|f(z)| < e^{(T+\epsilon)|z|}$$

for all sufficiently large $|z|$. The following three theorems were proved respectively by D. V. Widder [4], I. J. Schoenberg [2], and H. Poritsky [1] and J. M. Whittaker [3].

THEOREM 1. (Widder.) *If a real function $f(x)$, of class C^∞ in $0 \leq x \leq 1$, satisfies the condition*

$$(2) \quad (-1)^n f^{(2n)}(x) \geq 0, \quad 0 \leq x \leq 1; \quad n = 0, 1, 2, \dots,$$

then $f(x)$ coincides on $(0, 1)$ with an entire function of exponential type at most π .

THEOREM 2. (Schoenberg.) *If $f(z)$ is an entire function of exponential type at most T , and if*

$$(3) \quad f^{(2n)}(0) = f^{(2n)}(1) = 0, \quad n = 0, 1, 2, \dots,$$

then $f(z)$ is a sine polynomial of order at most T/π :

$$f(z) = \sum_{k=0}^N a_k \sin k\pi z, \quad N \leq T/\pi.$$

Let $\Lambda_n(z)$ be the polynomial of degree $2n+1$ determined by the relations

$$\begin{aligned} \Lambda_0(z) &= z; & \Lambda_n(0) &= \Lambda_n(1) = 0, & n &\geq 1; \\ \Lambda_n''(z) &= \Lambda_{n-1}(z), & n &\geq 1. \end{aligned}$$

THEOREM 3. (Poritsky-Whittaker.) *If $f(z)$ is an entire function of exponential type at most T , $T < \pi$, then $f(z)$ can be represented in the form*

¹ Presented to the Society, February 22, 1941.

² G. Valiron, *Lectures on the General Theory of Integral Functions*, Toulouse, 1923, p. 41f.

$$(4) \quad f(z) = \sum_{n=0}^{\infty} f^{(2n)}(1) \Lambda_n(z) - \sum_{n=0}^{\infty} f^{(2n)}(0) \Lambda_n(z-1),$$

where the series converge uniformly in any bounded region.

An expansion of the form (4) is called a Lidstone series.

The chief purpose of this note is to give a short proof of Theorem 1 which I obtained some time after Professor Widder communicated the theorem to me.³ I also give simple proofs of Theorems 2 and 3, suggested by the proof of Theorem 1.

LEMMA 1. If k and n are positive integers, and $f(x)$ is of class C^{2n} in $0 \leq x \leq 1$, then

$$(5) \quad \int_0^1 f(x) \sin k\pi x \, dx = \sum_{m=0}^{n-1} \frac{(-1)^m}{(k\pi)^{2m+1}} \{f^{(2m)}(0) - (-1)^k f^{(2m)}(1)\} \\ + \frac{(-1)^n}{(k\pi)^{2n}} \int_0^1 f^{(2n)}(x) \sin k\pi x \, dx.$$

This is proved by integrating the left-hand side repeatedly by parts.

LEMMA 2. If $g(x)$ is non-negative and concave in $a \leq x \leq b$, and

$$\int_a^b g(x) \, dx \leq A,$$

then

$$(6) \quad g(x) \leq \frac{2A}{b-a}, \quad a \leq x \leq b.$$

Let $g(x)$ take its maximum, G , at $x=x_0$. Since $g(x)$ is concave, the graph of $g(x)$ is above the broken line connecting the points $(a, 0)$ and $(b, 0)$ to (x_0, G) . The area under the broken line is $\frac{1}{2}G(b-a)$. Hence $\frac{1}{2}G(b-a) \leq A$, and (6) follows.

PROOF OF THEOREM 1. Take $k=1$ in Lemma 1. By hypothesis the terms of the sum on the right of (5) are all non-negative. Hence, for any positive integer n ,

$$B = \int_0^1 f(x) \sin \pi x \, dx \geq \frac{1}{\pi^{2n}} \int_0^1 (-1)^n f^{(2n)}(x) \sin \pi x \, dx \geq 0,$$

$$\int_0^{1-\delta} (-1)^n f^{(2n)}(x) \sin \pi x \, dx \leq B\pi^{2n}, \quad 0 < \delta < \frac{1}{2},$$

³ (Added in proof.) Essentially the same proof was found independently by Professor Schoenberg.

$$\int_{\delta}^{1-\delta} (-1)^n f^{(2n)}(x) dx \leq \frac{B\pi^{2n}}{\sin \pi\delta}.$$

Since $(-1)^n f^{(2n+2)}(x) \leq 0$, $(-1)^n f^{(2n)}(x)$ is non-negative and concave in $(\delta, 1-\delta)$. By Lemma 2,

$$(7) \quad 0 \leq (-1)^n f^{(2n)}(x) \leq \frac{2B\pi^{2n}}{(1-2\delta) \sin \pi\delta}, \quad \delta \leq x \leq 1-\delta.$$

Let $h = \frac{1}{2} - \delta$. For any x in $(\delta, 1-\delta)$, Taylor's theorem with remainder of order 2, applied to $f^{(2n)}(x)$, leads to the relation

$$f^{(2n+1)}(x) = \pm \frac{1}{h} [f^{(2n)}(x \pm h) - f^{(2n)}(x)] \mp \frac{1}{2} h f^{(2n+2)}(x \pm \theta h),$$

$0 < \theta < 1,$

where the upper signs or the lower signs are taken according as $x+h$ or $x-h$ is in $(\delta, 1-\delta)$. Using (7), written both for n and $n+1$, we obtain

$$|f^{(2n+1)}(x)| \leq \left(\frac{2}{h} + \frac{\pi^2 h}{2} \right) \frac{2B\pi^{2n}}{(1-2\delta) \sin \pi\delta} = C(\delta) B\pi^{2n+1},$$

where $C(\delta)$ depends only on δ . Thus

$$(8) \quad \limsup_{n \rightarrow \infty} |f^{(n)}(x)|^{1/n} \leq \pi,$$

uniformly in any interval $\delta < x < 1-\delta$, where $0 < \delta < \frac{1}{2}$. The relation (8) implies first that the Taylor series of $f(x)$ about any point in $(0, 1)$ converges to an entire function coinciding with $f(x)$ in $(0, 1)$; (8) then shows that this function is of exponential type at most π .

PROOF OF THEOREM 2. Since $f(x)$ is of class C^∞ , it is represented in $(0, 1)$ by its Fourier sine series:

$$(9) \quad f(x) = \sum_{k=0}^{\infty} a_k \sin k\pi x, \quad 0 < x < 1,$$

where

$$a_k = 2 \int_0^1 f(t) \sin k\pi t dt.$$

Since (3) is true, Lemma 1 shows that

$$a_k = \frac{2(-1)^n}{(k\pi)^{2n}} \int_0^1 f^{(2n)}(t) \sin k\pi t dt$$

for every positive integer n . Hence

$$(10) \quad |a_k| \leq \frac{2}{(k\pi)^{2n}} \max_{0 \leq x \leq 1} |f^{(2n)}(x)|.$$

If $k\pi > T$, for large n we have

$$\max_{0 \leq x \leq 1} |f^{(2n)}(x)| \leq S^{2n}, \quad T < S < k\pi,$$

since $f(x)$ is of exponential type at most T . From (10) it then follows that $a_k = 0$ if $k\pi > T$. Thus all terms of (9) with $k > T/\pi$ vanish, and Theorem 2 is proved.

PROOF OF THEOREM 3. The function $f(x)$ is represented in $(0, 1)$ by the Fourier series (9), and $\frac{1}{2}a_k$ is just the integral on the left of (5). Hence, for every positive integer n ,

$$(11) \quad \begin{aligned} f(x) = & \sum_{m=0}^{n-1} f^{(2m)}(1) \sum_{k=1}^{\infty} \frac{2(-1)^{k+m+1} \sin k\pi x}{(k\pi)^{2m+1}} \\ & - \sum_{m=0}^{n-1} f^{(2m)}(0) \sum_{k=1}^{\infty} \frac{2(-1)^{m+1} \sin k\pi x}{(k\pi)^{2m+1}} + \frac{1}{2} R_n(x), \end{aligned}$$

where

$$R_n(x) = (-1)^n \int_0^1 f^{(2n)}(t) \sum_{k=1}^{\infty} \frac{\sin k\pi x \sin k\pi t}{(k\pi)^{2n}} dt.$$

The infinite series appearing in (11) are the Fourier sine series of $\Lambda_m(x)$ and $\Lambda_m(1-x)$, as given by Whittaker;⁴ they can easily be checked by successive integrations of the well known Fourier sine series of $\Lambda_0(x) = x$. Relation (11) becomes

$$f(x) = \sum_{m=0}^{n-1} f^{(2m)}(1) \Lambda_m(x) - \sum_{m=0}^{n-1} f^{(2m)}(0) \Lambda_m(x-1) + R_n(x).$$

We have thus obtained a "Lidstone series with remainder," with an expression for the remainder as a real integral equivalent to that given by Widder [4].

We have

$$|R_n(x)| \leq \frac{1}{\pi^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \int_0^1 |f^{(2n)}(t)| dt,$$

and this approaches zero as $n \rightarrow \infty$, uniformly in $(0, 1)$, if $f(x)$ satisfies (1) with $T < \pi$. Hence under the hypothesis of Theorem 3 the series in (4) converges to $f(x)$ uniformly in $(0, 1)$. Now Whittaker has

⁴ [3, p. 454].

shown⁵ that if a Lidstone series converges for some non-integral value of z , it converges for all z , uniformly in any bounded region, and so represents an entire function, which in our case must be $f(z)$. This completes the proof of Theorem 3.

By applying, instead of Lemma 1, the formula obtained by integrating

$$\int_0^1 f(x) \cos \frac{1}{2} k \pi x \, dx$$

by parts, we can prove Schoenberg's theorem [2], analogous to Theorem 2, that a function $f(z)$ of exponential type is a cosine polynomial if $f^{(2n)}(1) = f^{(2n+1)}(0) = 0$ ($n = 0, 1, 2, \dots$); and we can obtain Whittaker's result [3] corresponding to Theorem 3, concerning the expansion of $f(z)$ in a series with coefficients $f^{(2n)}(1), f^{(2n+1)}(0)$. The analogue of Theorem 1 is

THEOREM 4. *If $f(x)$ is of class C^∞ in $0 \leq x \leq 1$, and*

$$f^{(4n)}(x) \geq 0, \quad (-1)^n f^{(2n)}(1) \geq 0, \quad (-1)^n f^{(2n+1)}(0) \leq 0, \\ n = 0, 1, 2, \dots,$$

then $f(x)$ coincides over $(0, 1)$ with an entire function of exponential type at most $\pi/2$.

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⁵ [3, p. 455].

REPRESENTATIONS OF BOOLEAN ALGEBRAS

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There are several proofs in the literature¹ of M. H. Stone's theorem on the representation of Boolean algebras by sets [2, 4, 5, 7, 8, 9]. This note contains a simplified version of Stone's original proof, adapted to the following set, I-IV, of postulates for a Boolean algebra B in terms of the special element 0 and the undefined operations *product* ab and *negation* b' . It is assumed that 0 is in B , and that if a , b , and c are in B , then ab and b' are in B , and

$$\text{I. } ab = ba.$$

$$\text{II. } a(bc) = (ab)c.$$

$$\text{III. } aa = a.$$

$$\text{IV. } ab = a \text{ if and only if } ab' = 0.$$

Replacing b by a in IV gives V: $aa' = 0$. Since $a0 = a(aa') = (aa)a' = aa' = 0$, we have VI: $a0 = 0$.

DEFINITIONS. A point is a set P of elements of B such that

α . The element 0 is not in P .

β . If a is in P and b is in P , then ab is in P .

γ . P is maximal with respect to properties α and β .

The set R_a of all points P which contain a is defined to be the representative set corresponding to the element a .

LEMMA 1. If ab is in P , then a is in P .

PROOF. If a were not in P , then P would not be maximal, since a and all products pa , where p is in P , could then be added to P without disturbing α , since if $pa = 0$, then $pab = 0$.

LEMMA 2. If a is not equal to 0 , then a is in some point P .

PROOF. All sets of elements of B which contain a and satisfy α and β form a system S partially ordered by set inclusion. Any linearly ordered subsystem L of S has an upper bound in S , namely the union of all members of L . Hence by Zorn's lemma [10, 11], there exists in S at least one maximal element P .

THEOREM. The correspondence between elements a of B and their representative sets R_a is an isomorphism; that is, 1. $R_{ab} = R_a \cap R_b$; 2. $R_{a'} = C(R_a)$; 3. if $R_a = R_b$, then $a = b$.

¹ See also N. H. McCoy and D. Montgomery, Duke Mathematical Journal, vol. 3 (1937), pp. 455-459.

PROOF. 1. If P is in R_a and R_b , then it is in R_{ab} by β . Conversely, if P is in R_{ab} , it is in R_a and R_b by Lemma 1.

2. R_a and $R_{a'}$ are complementary, for if a point P is not in $R_{a'}$, there is an element b in P such that $a'b=0$, since otherwise a' and products $a'p$ could be added to P , and P would not be maximal. Hence by IV, $ab=b$; therefore ab is in P . Then by Lemma 1, a is in P , and P is in R_a . On the other hand, R_a and $R_{a'}$ have no common point P , since such a P would have to contain $aa'=0$.

3. If $a \neq b$, then either $ab' \neq 0$ or $a'b \neq 0$, since otherwise by IV $a=ab=b$. If $ab' \neq 0$, then by Lemma 2, ab' is in some point P . By Lemma 1, a is in P . But b is not in P , since $ab'b=0$. Likewise if $a'b \neq 0$, there is a point containing b but not a . Hence $R_a \neq R_b$.

COROLLARY. *The set I-IV is an adequate postulate system for Boolean algebras.*

For I-IV hold in any Boolean algebra, and any algebra in which they hold has been shown to be isomorphic to an algebra of sets, and hence to a Boolean algebra. This postulate system is comparable in simplicity with other well known sets [1, 3, 6].

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A FIXED-POINT THEOREM FOR TREES¹

A. D. WALLACE

By a *tree* we mean a compact (= bicomact) Hausdorff space which is acyclic in the sense that

(i) if $\mathcal{U}$ is a f.o.c. (=finite open covering) of a tree T then there is a f.o.c. $\mathcal{B} \subset \mathcal{U}$ such that the nerve $N(\mathcal{B})$ is a combinatorial tree,

and which is locally connected in the sense that

(ii) if $\mathcal{U}$ is a f.o.c. of T then there is a f.o.c. $\mathcal{B} \subset \mathcal{U}$ whose vertices are connected sets.

It may be shown [3] that an acyclic continuous curve in the usual sense is a tree in our terminology. If q is a mapping which assigns to each point t of a topological space a set qt in a topological space, then we say that q is *continuous* provided that for each t and each neighborhood U of qt we can find an open set V containing t such that if t' is in V then qt' is in U . Our present purpose is to establish the following result:

(A) Let T be a tree and let q be a continuous point-to-set mapping which assigns to each point t a continuum qt in T . Then there is a $t_0 \in T$ such that $t_0 \in qt_0$.

The proof (which is divided into several lemmas) uses strongly a technique introduced by H. Hopf [1]. However the present note has been made self-contained.

(A₁) The intersection of two continua of T is again a continuum.

PROOF. Let B_1, B_2 be two continua such that $B_1 \cdot B_2 = C_1 + C_2$ where the C_i are disjoint and closed. We can find disjoint open sets $D_i \supset C_i$. Let $t \in T - B_1 \cdot B_2$. We can then find an open set V_t containing t and which does not meet both B_1 and B_2 . The sets D_i together with the sets V_t can be reduced to a f.o.c. $\mathcal{U}$ of T . Let $\mathcal{B} \subset \mathcal{U}$ be the f.o.c. described in (i). Let $\mathcal{B}_i$ be those vertices of $\mathcal{B}$ on B_i . It is easy to see that $N(\mathcal{B}_i)$ is connected. If $c_j \in C_j$ we can find a chain of 1-cells E_i in $N(\mathcal{B}_i)$ whose first vertex contains c_1 and whose last vertex contains c_2 . Now we cannot have $E_i \subset D_1 + D_2$ and E_i contains a vertex which is not on B_j . Hence $E_1 \neq E_2$ and so $N(\mathcal{B})$ is not a tree. This contradiction completes the proof.

¹ Presented to the Society, May 3, 1941.

(A₂) Any f.o.c. $\mathcal{U}$ of T contains a f.c.c. $\mathfrak{F} \subset \mathcal{U}$ so that each $F_i \in \mathfrak{F}$ is connected and further $N(\mathfrak{F})$ is a combinatorial tree.

PROOF. We can find a f.o.c. $\mathfrak{B} \subset \mathcal{U}$ such that $N(\mathfrak{B})$ is a tree. By a lemma due to Čech [5, p. 180] we can find a f.c.c. $\mathfrak{F}' \subset \mathfrak{B}$ such that $\mathfrak{F}'$ and $\mathfrak{B}$ are combinatorially isomorphic. Let $\mathfrak{N}_i$ be the f.o.c. $(V_i, T - F'_i)$. Using (ii) it is easy to see that there is a f.o.c. $\mathfrak{W}$ such that each W_i is connected and $\mathfrak{W} \subset \mathfrak{N}_i$, for each i . Let i be fixed. If W_j meets F'_i then so does $\overline{W}_j$ and so is contained in V_i . Let Q_i be the union of all such W_j . Then the closure of this set has a component-wise decomposition, say $\overline{Q}_i = F_{i1} + F_{i2} + \cdots + F_{ir}$. Let $\mathfrak{F}$ be the f.c.c. $\{F_{ij}\}$. It is clear that the elements of $\mathfrak{F}$ are connected and it is not hard to show that $\dim \mathfrak{F} \leq 1$, that is, at most two elements of $\mathfrak{F}$ have a non-null intersection. If we have a chain

$$F_{i_1 j_1}, F_{i_2 j_2}, \dots, F_{i_r j_r}, F_{i_1 j_1} \quad r > 2,$$

such that each set meets the following but such that there are no other intersections, then the sets $F_{i_1 j_1}$ and $\sum_{i > 1} F_{i j_i}$ are connected and therefore by (A₁) so is their meet, the set $F_{i_1 j_1} \cdot F_{i_2 j_2} + F_{i_1 j_1} \cdot F_{i_r j_r}$. But then we would have $F_{i_1 j_1} \cdot F_{i_2 j_2} \cdot F_{i_r j_r} \neq 0$, a contradiction. It follows that $N(\mathfrak{F})$ is a tree.

(B) Let q be a mapping which assigns to each continuum K in T a continuum qK in T such that if $K_1 \subset K_2$, then $qK_1 \subset qK_2$. If $\mathfrak{F} = \{F_i\}$ is a f.c.c. with connected sets such that $N(\mathfrak{F})$ is a tree then there is an F_i for which $F_i \cdot qF_i \neq 0$.

PROOF. Let $N = N(\mathfrak{F})$ and suppose that the vertices of N are e_i . To each i we assign an i' so that $F_{i'}$ meets qF_i . We then have a mapping $e_i \rightarrow e_{i'}$ and since N is a tree it follows at once by a result due to Hopf [1, Lemma γ] that we can find an edge $e_m e_n$ which is contained in the chain joining e_m to e_n .² We show that $F_k \cdot qF_k \neq 0$, $k = m$, or n . We have $F_m \cdot F_n \neq 0$ and by construction $F_m \cdot qF_m \neq 0 \neq F_n \cdot qF_n$. Further

$$(*) \quad F_{m'}, F_i, \dots, F_m, F_n, F_j, \dots, F_{n'}$$

is a simple chain of sets. Of course it may happen that F_n precedes F_m in (*) but this is of no importance. Let X be the union of all the sets in (*) from $F_{m'}$ up to and including F_m . Let Y be similarly defined for the other part of (*). Then X and Y are continua with $X \cdot Y = F_m \cdot F_n$.

² I am indebted to Professor S. Lefschetz for the remark that $e_i \rightarrow e_{i'}$ generates a chain-mapping (that is, a mapping permutable with the boundary operator) if we define for the image of $e_m e_n$ the chain joining e_m to e_n . Since N is acyclic it follows at once that there is a fixed element. This may replace the result of Hopf.

Also $F_m + F_n$ is a continuum and so is $Z = qF_m + qF_n$. Clearly Z meets the end-vertices of $(*)$. By (A_1) $Z \cdot (X + Y)$ is a continuum. Hence $Z \cdot X \cdot Y$ is not null. Thus $F_m \cdot F_n \cdot (qF_m + qF_n) \neq 0$ and this completes the proof of (B).

It is not hard to see that if q is a mapping of the type described in (A) then q satisfies the conditions in (B) if we define $qK = \sum qt, t \in K$, for each continuum K of T . The proof is quite similar to those for analogous results concerning single-valued mappings.

We now turn to a proof of (A). Suppose that no t is in qt . We can find a neighborhood R_t of t so that $\bar{R}_t$ does not meet qt . Let $V_t = T - \bar{R}_t$. Since $qt \subset V_t$ we can find a neighborhood S_t of t so that $t' \in S_t$ implies $qt' \subset V_t$. Let U_t be the meet of R_t and S_t . We cover T by a finite sub-collection $\{U_i\} = \{U_{t_i}\}$ of the sets U_t . We can find a refinement $\mathfrak{F}$ of $\mathfrak{U} = \{U_i\}$ which satisfies the conditions in (B) in consequence of (A_2) . By (B) we can find a set F in $\mathfrak{F}$ so that F meets qF . In other words we find a t in F such that F meets qt . Now F is in some U_i and hence qt is in the corresponding V_i . But since F does not meet the set V_i it cannot meet qt . This contradiction completes the proof.

A continuous transformation $fM = N$ is said to be *free* (Hopf [1]) provided there is a continuous transformation $gM \subset M$ such that $fgx \neq fx$ for each $x \in M$. The transformation f is *monotone* if the set $f^{-1}y$ is connected for each $y \in N$.

(C) *No continuum admits a free monotone transformation onto a tree.*

PROOF. Let $fM = T$ be monotone and $gM \subset M$ be continuous. For each $t \in T$ we set $qt = fgf^{-1}t$. It is not hard to see that q is continuous and hence we may apply (A). But from $t \in qt$ it follows at once that there is an $x \in M$ with $fgx = fx$.

The transformations $fM \subset N$ and $gM \subset N$ have a *coincidence* (Lefschetz [2]) if there is an $x \in M$ with $fx = gx$. As in (C) we may show that

(D) *A monotone transformation $fM = T$ of a continuum onto a tree admits a coincidence with any continuous transformation $gM \subset T$.*

Remarks. The result (A) is usually called the Scherrer fixed-point theorem when q is single-valued and T is an acyclic continuous curve. For a list of papers concerning it see Hopf [1]. Corollary (C) will be found in [3]. The result (A) was found while constructing a proof of (D). Finally (A) is analogous to a result of S. Kakutani [4] who has shown that if S is an n -simplex and to each $s \in S$ we assign continuously a closed convex set qs then there is an $s_0 \in qs_0$.

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ON THE DEFINITION OF CONTACT TRANSFORMATIONS

ALEXANDER OSTROWSKI

If z is a function of $x_1, \dots, x_n$ and $p_\nu = \partial z / \partial x_\nu$, $\nu = 1, \dots, n$, a *contact transformation* in the space of $z, x_1, \dots, x_n$, is defined by a set of $n+1$ equations

$$(a) \quad Z = Z(z, x_\mu, p_\mu), \quad X_\nu = X_\nu(z, x_\mu, p_\mu), \quad \nu = 1, \dots, n,$$

such that *firstly* in calculating the n derivatives

$$P_\nu = \frac{\partial Z}{\partial X_\nu}, \quad \nu = 1, \dots, n,$$

the expressions for the P_ν are given by a set of n equations

$$(b) \quad P_\nu = P_\nu(z, x_\mu, p_\mu), \quad \nu = 1, \dots, n,$$

in which the derivatives of the p_μ *fall out*; and *secondly* the equations (a) and (b) can be resolved with respect to z, x_μ, p_μ :

$$(A) \quad z = z(Z, X_\mu, P_\mu), \quad x_\nu = x_\nu(Z, X_\mu, P_\mu), \quad \nu = 1, \dots, n,$$

$$(B) \quad p_\nu = p_\nu(Z, X_\mu, P_\mu), \quad \nu = 1, \dots, n.$$

These two postulates are equivalent with the hypothesis that the $2n+1$ equations (a), (b) form a transformation between the two spaces of the sets of $2n+1$ independent variables (z, x_ν, p_ν) , (Z, X_ν, P_ν) satisfying the Pfaffian condition

$$dZ - \sum_{\nu=1}^n P_\nu dX_\nu = \rho \left(dz - \sum_{\nu=1}^n p_\nu dx_\nu \right), \quad \rho \neq 0.$$

In the following lines we prove: *the hypothesis that the system (A) is a corollary of the system (a) and conversely is already sufficient in order that (a) define a contact transformation*, that is to say: under this hypothesis the expressions (b) of P_ν , derived from (a), are independent of the second derivatives of z .

As to the functions $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$, $z(Z, X_\mu, P_\mu)$, $x_\nu(Z, X_\mu, P_\mu)$, we shall assume:

(1) that the functions $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$ possess continuous partial derivatives of the first order with respect to their $2n+1$ arguments;

(2) that the "total Jacobian"

$$(1) \quad \left| \frac{dX_\nu}{dx_\mu} \right|, \quad \nu, \mu = 1, \dots, n,$$

does not vanish identically in the $(2n+1) + n(n+1)/2$ variables $z, x_\nu, p_\nu, p'_{x_\mu}$. Here the "total derivative" with respect to x_ν is defined by

$$(2) \quad \frac{d}{dx_\nu} = p_\nu \frac{\partial}{\partial z} + \frac{\partial}{\partial x_\nu} + \sum_{\mu=1}^n p'_{\mu x_\nu} \frac{\partial}{\partial p_\mu}, \quad \nu = 1, \dots, n;$$

(3) that the functions $z(Z, X_\mu, P_\mu)$, $x_\nu(Z, X_\mu, P_\mu)$ possess continuous partial derivatives of the first order with respect to their $2n+1$ arguments. (This hypothesis is certainly satisfied if the functions $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$ possess continuous partial derivatives of the *second* order with respect to their arguments and if the determinant (1) does not vanish.)

From these three hypotheses it follows at once that the determinant $|dx_\nu/dX_\mu|$, $\nu, \mu = 1, \dots, n$, does not vanish identically, since $x_1, \dots, x_n$ can be assumed as being independent variables.

Then, if $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$ were all free of the p_μ , we have obviously a reversible point-to-point transformation between the space of $n+1$ variables (z, x_ν) and that of $n+1$ variables (Z, X_ν) . And the same result holds if $z(Z, X_\mu, P_\mu)$, $x_\nu(Z, X_\mu, P_\mu)$ were all free of the P_μ . We may therefore assume without loss of generality that p_μ do actually appear in the equations (a) and P_μ in the equations (A).

By means of total derivatives (2), P_μ can be calculated from the n equations

$$(3) \quad \frac{dZ}{dx_\nu} = \sum_{\mu=1}^n P_\mu \frac{dX_\mu}{dx_\nu}, \quad \nu = 1, \dots, n.$$

Consider the n expressions

$$(4) \quad B_\nu = \frac{\partial Z}{\partial p_\nu} - \sum_{\mu=1}^n P_\mu \frac{\partial X_\mu}{\partial p_\nu}, \quad \nu = 1, \dots, n,$$

and suppose first that not all B_ν vanish.

Then, if for instance $B_1 \neq 0$, let

$$q_\lambda = \frac{\partial p_\lambda}{\partial x_1} = \frac{\partial p_1}{\partial x_\lambda}, \quad \lambda = 1, \dots, n.$$

In differentiating (3) with respect to q_λ we have easily

$$\sum_{\mu=1}^n P'_\mu \frac{dX_\mu}{dx_\nu} = \delta_\nu^\lambda B_1 + \delta_\nu^1 (1 - \delta_\lambda^1) B_\lambda, \quad \nu, \lambda = 1, \dots, n,$$

where as usual

$$\delta_\nu^\mu = \begin{cases} 0, & \mu \neq \nu, \\ 1, & \mu = \nu. \end{cases}$$

But now it follows that

$$\frac{\partial(P_1, \dots, P_n)}{\partial(p'_{1x_1}, \dots, p'_{nx_1})} \left| \frac{dX_\mu}{dx_\nu} \right| = |\delta_\nu^\lambda B_1 + \delta_\nu^1 (1 - \delta_\lambda^1) B_\lambda| = B_1^n \neq 0,$$

the P_ν are independent with respect to $p'_{1x_1}, \dots, p'_{nx_1}$, and the equations (A) are only possible, if they do not contain the P_ν at all, the case which has been already discarded.

We have therefore $B_\nu = 0$, $\nu = 1, \dots, n$. Then the equations (3) and (4) reduce to the $2n$ equations

$$(5) \quad \begin{aligned} \frac{\partial Z}{\partial x_\nu} + p_\nu \frac{\partial Z}{\partial z} &= \sum_{\mu=1}^n P_\mu \left(\frac{\partial X_\mu}{\partial x_\nu} + p_\nu \frac{\partial X_\mu}{\partial z} \right), & \nu = 1, \dots, n, \\ \frac{\partial Z}{\partial p_\nu} &= \sum_{\mu=1}^n P_\mu \frac{\partial X_\mu}{\partial p_\nu}, & \nu = 1, \dots, n. \end{aligned}$$

On the other hand, the rank of the matrix with n columns and $2n$ rows

$$\begin{pmatrix} \frac{\partial X_\mu}{\partial x_\nu} + p_\nu \frac{\partial X_\mu}{\partial z} \\ \frac{\partial X_\mu}{\partial p_\nu} \end{pmatrix}, \quad \mu, \nu = 1, \dots, n,$$

is n , since otherwise (1) would vanish. We see that in this case P_ν can be expressed from (5) by z, x_μ, p_μ .

Since the same argument applies to the equations (A), p_ν can be expressed by means of Z , X_μ , P_μ .

We have now the 4 sets of relations (a), (b), (A), (B). It is easily seen that the $2n+1$ relations (A), (B) are inverse of the $2n+1$ relations (a), (b), if p_μ resp. P_μ are considered as independent variables. Indeed, in putting the values (a) and (b) in the relations (A), (B), we must obtain identities $z=z$, $x_\nu=x_\nu$, $p_\nu=p_\nu$, for otherwise a non-identical relation between z , x_μ , p_μ would follow, that is, a differential equation, satisfied by an "arbitrary" function $z(x_1, \dots, x_n)$.

We see that in the case of *one* function of n variables a reversible transformation of the first order is necessarily a contact transformation.

Our implicit definition of the "reversible transformations of the first order" leads to non-trivial results in the cases in which the contact transformations in the usual sense do not exist at all. For instance, in the case of $n > 1$ functions $z_1(x), \dots, z_n(x)$ of one independent variable, all contact transformations reduce simply to the point-to-point transformations in the space of $n+1$ variables $z_1, \dots, z_n, x$. On the other hand, there exist in this case non-trivial reversible transformations. If for instance

$$X = z_n - x \sum_{\nu=1}^n p_\nu, \quad Z_\lambda = z_\lambda, \quad (\lambda = 1, \dots, n-1), \quad Z_n = - \sum_{\nu=1}^n p_\nu,$$

$$p_\nu = \frac{dz_\nu}{dx}, \quad P_\nu = \frac{dZ_\nu}{dX}, \quad \nu = 1, \dots, n,$$

we have easily for $\lambda = 1, \dots, n-1$

$$\frac{P_\lambda}{xP_n - 1} = \frac{p_\lambda}{\sum_{\kappa=1}^{n-1} p_\kappa}, \quad x = \frac{1 + \sum_{\lambda=1}^{n-1} P_\lambda}{P_n},$$

and therefore

$$z_n = X - \frac{Z_n}{P_n} \left(1 + \sum_{\lambda=1}^{n-1} P_\lambda \right), \quad z_\lambda = Z_\lambda, \quad \lambda = 1, \dots, n-1.$$

We have determined in the case of n functions of one variable all reversible transformations of the first order by means of certain Pfaffian and Mongeian relations. These results will be exposed in another paper.

FOURTEEN SPECIES OF SKEW HEXAGONS

H. S. WHITE

1. **Hexagon and hexahedron.** For a tentative definition, let a skew hexagon be a succession of six line segments or edges, finite or infinite, the terminal point of each being also the initial point of another, the terminus of the last being the first initial point. Secondly, let there let be six marks to designate the points, the six to be in a fixed order. Further, let the first point be any one of the six, so that only the cyclic order is essential. At each of the six points, pass a plane through the two lines, and denote the plane by the same mark as that point. Adjoin to the figure these six entire planes, a complete hexahedron.

2. **Sets of 64 hexagons.** The initial and terminal points of an edge denote ambiguously two segments, one finite and one infinite. If the marks in their order are 123456, then that cycle may denote equally any one of 64 different hexagons, $2^6 = 64$, obtained by selection. If desired, the one whose edges are all finite may be taken to represent the set. But we can define positive and negative directions along each line, arbitrarily on one, then by inference on all other lines of intersection of the planes. Thereafter it will be natural to understand by 12 a positive segment on the line, by 21 a negative.

3. **Hexagons and cyclic permutations.** As above defined, every skew hexagon determines one and only one set of six planes, but these intersect, not in 6 but in 15 lines, and so contain many different hexagons. Any cyclic permutation of the marks of the planes, as 154263; may be interpreted as describing a hexagon. An edge is given by any four consecutive marks; for example, 2631 will denote a segment on the intersection-line 63, from its point in plane 2 to its point in plane 1. Thus a cycle reversed will denote the same edges in reverse order and we shall consider the two as identical. As there are 120 cyclic permutations of 123456, this gives us only 60 different hexagons to assort into species.

All sets of 6 planes will be defined as equivalent, provided no four have a common point. By moving planes freely any one can be made to coincide with any other set, without passing through the excluded positions. It is well known that the traces of five planes in a sixth bound in that sixth one convex pentagon as well as certain triangles and quadrilaterals. Of the six pentagons, each has two edges which are edges of other pentagons, thus providing a natural order of the planes. Moreover those six common edges form a continuous

broken line or skew hexagon, which shall be called the *normal hexagon* of that set of planes. Let its vertices and planes be marked, as described above, in their order along the normal hexagon, 1, 2, 3, 4, 5, 6, the vertex 1 being any one taken at random. This gives 6 equivalent notations for the set; in other words, *the set of 6 planes is equivalent to itself under the cyclic permutations of the normal hexagon*. Consequently, any other skew hexagon occurring in the figure is equivalent to that derived from it by cyclic permutation: $S = (123456)$, of the marks of its 6 planes and vertices. Note that the planes take their marks from the vertices of the normal hexagon, after which the vertices of any other hexagon in the figure (having two adjacent edges in each plane) take their marks each from the plane that contains the two edges at that point.

For example, 146235 will denote a skew hexagon different from the normal. The operations repeated give the 5 equivalent cycles:

2	5	1	3	4	6,
3	6	2	4	5	1,
4	1	3	5	6	2,
5	2	4	6	1	3,
6	3	5	1	2	4.

Thus there are found sets of 6 equivalent hexagons, but also some of fewer than 6. For example 153426 is in a set of only 3.

4. **Survey of all species of skew hexagons.** On any single set of six planes will be found all possible species of hexagons, for any given

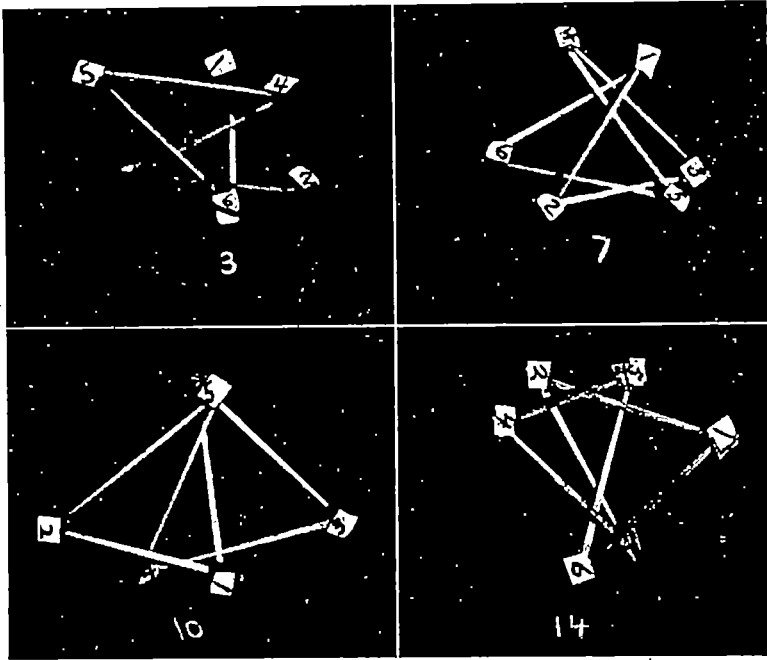
TABLE I

No.	In Set	Cycle	No.	In Set	Cycle
1	1	123456	8	6	132645
2	6	146235	9	3	123654
3	6	132654	10	3	135264
4	6	125364	11	3	135462
5	6	142356	12	3	132564
6	6	143562	13	3	142365
7	6	162345	14	2	143652

hexagon, nonsingular, gives rise to a set of 6 planes, and all such sets are equivalent. If the planes have been referred to their normal hexagon, then all 60 hexagons that lie in all six planes are represented in Table 1 above, a single cycle representing six, or three, or two types

of 64 that share a single set of 6 lines, or the one unique normal type.

5. **A positive order on each line.** Suppose the vertices of the normal hexagon to be marked in the order of their occurrence along its periphery: 123456. Each segment, as 12, is known to be tangent at some



EXPERIMENTAL MODELS, CRUDE BUT SERVICEABLE,
OF SKEW HEXAGONS

No. 3, Normalized cycle? 143265.

No. 7, Normalized cycle? 164532.

No. 10, Normalized cycle? 146253.

No. 14, Normalized cycle? 125634.

one of its points to a conic enveloping, in plane 1, the traces of the other planes in the order, which shall be chosen as *positive* order, of their arrangement about the unique convex pentagon formed by those traces, say 23456, and indeed it touches on the same segment the line conics in planes 1 and 2. Then on any other line, as 5, in plane 1 the intersections of the others must occur in the same order, 2346, omitting its own mark 5. So on all the lines in plane 1 we have a

"normal" assignment of marks. Then since each line lies in two planes, the marking is automatically extended to all the lines in all 6 planes.

6. **A character of every hexagon.** On the lines of the normal hexagon the external segment must contain two intersections of other

TABLE II

Species	Character	Species	Character
1	000000	8	120102
2	002220	9	221001
3	221102	10	001221
4	001210	11	110110
5	000112	12	211101
6	010201	13	101112
7	001010	14	020202

planes. And since the internal segment is designated by terminals in their positive order, the external will be negative from the same initial point to the same terminus. Give it therefore two marks—and 2. Extending this scheme, assign to each directed segment a sign + or —, and either 0, 1, or 2 to indicate how many planes cut it internally. Every hexagon will then have six double "characters" in a definite cyclic order, the normal hexagon being described by $(0^+ 0^+ 0^+ 0^+ 0^+ 0^+)$. We have now to list the characters of all 14 species for comparison.

A segment of such a hexagon being given by an ordered tetrad, as 4265, we have yet to choose between the positive and the negative direction from 4 to 5. *Always we shall take the positives for a typical representative of the 64 which share the same designation* (cf. §4). With this convention, we need write only the character 0, 1, or 2 for each segment of the hexagon. In the above example, on the line 26 the marks 2 and 6 are not in intersecting planes, and the two intersections 1, 3 do not fall on the positive segment from 4 to 5. The mark will be 0. Again, for a tetrad 2615 it is 2, for 6413 it is 1, and so on. For the 14 species the characters of the representative hexads are shown in Table II.

These characters are not quite distinctive, but nearly so. Sets 3 and 5 differ as right-hand and left-; also sets 12 and 13. But not so sets 8, 9, 10, and 14: each is auto-symmetric, and a change of sense will not exchange them. Distinguish odd and even triads of planes, and we see that triads are collected in No. 10, alternated in sequence in No. 9.

There are, therefore, 14 types among these 60 skew hexagons, all lying

in the same set of 6 planes, two segments in each plane. No two types are similar as regards distribution of intersection-points; but two pairs are symmetrical to each other, and four sets are auto-symmetric.

THEOREM. *The character of every skew hexagon must contain no aces, or two, or four. (A synthetic proof seems desirable.)*

7. Determining the species of a concrete model. When a given material model is presented for classification, one first assigns marks, as 123456 to the vertices in their order along the perimeter, first point and direction being chosen at random. Next, the cyclic order of points on each side is found by direct observation. From this, the determination of a *normal order* is most readily shown by an example.

From a certain specimen we take these sequences:

<i>On line</i>	<i>Order</i>	<i>On line</i>	<i>Order</i>
12	6453 or 6354	45	3162 or 3261
23	1654 or 1456	56	4321 or 1234
34	2615 or 2516	61	5423 or 5324

To unite these consistently in a hexad, take the first and third. They require 264153 or 261453. Comparison with the order on 23 decides in favor of 261453. Adopt this for *normal order*. Apply the substitution:

$$T = \begin{pmatrix} 261453 \\ 123456 \end{pmatrix},$$

by which the normal order becomes 123456. This redesignates the points, and the given hexagon is now entitled $T(123456) = 316452$. This has the *character* 202121. As this is not found in Table II, adopt the reverse normal order, and find for character 020101. This is No. 6 on the list. Verify by applying the operator $S = (123456)$ five times to 316452. The last member of the set is 265341, readily recognizable as the reverse of No. 6 which was 143562. As errors are easy in calculating the character, such a final check should always be demanded. Another example, if desired for practice, is taken from a second model. The orders observed are: 6543, 5146, 2156, 3612, 4132, 5324, leading to No. 8 in the table.

This plan of classification is hardly practicable for skew n -gons when $n > 6$, owing to the rapid increase in number of species. For a single species of n -gon, whose vertices are on a twisted cubic curve, it could be generalized.

ON THE EXISTENCE OF ELECTRICAL NETWORKS

C. E. CLARK

This paper is concerned with the existence of electrical networks which satisfy certain preassigned conditions. These conditions have to do with the existence of circuits with preassigned resistances in common.

Consider a finite set of points in euclidean 3-space and a set of straight line segments joining pairs of these points. Furthermore, suppose that no two of the segments intersect at an interior point, but any number of segments may have a common end-point. Each of the segments is called a *branch* of the graph. With each branch of the graph let there be associated a non-negative real number called the *resistance* of the branch. The graph together with the resistances is an *electrical network*. A *circuit* of a network is a topological circle of the network together with an orientation of this circle.

Let two circuits C_i and C_j have the branches b_p^p , $p=1, 2, \dots$, in common. Let r_p^p be the resistance of b_p^p . Let $\bar{r}_p^p = r_p^p$ if the orientations of C_i and C_j agree along b_p^p while $\bar{r}_p^p = -r_p^p$ if the orientations are opposite. Then $I_{ij} = \sum_p \bar{r}_p^p$, $p=1, 2, \dots$, is the *intersection* of C_i and C_j . We see that I_{ii} is the sum of the resistances of the branches of C_i .

Let C_i , $i=1, 2, \dots, n$, be distinct circuits of a network. Then the matrix $\|I_{ij}\|$, $i, j=1, \dots, n$, is the *intersection matrix* of the C_i . This matrix is symmetric and has non-negative diagonal elements. Any matrix with these two properties is an *intersection matrix*. An intersection matrix M is *realizable* when there exists a network which has a set of circuits whose intersection matrix is M .

THEOREM 1. *Given an intersection matrix*

$$(1) \qquad \|I_{ij}\|, \qquad i, j = 1, \dots, n;$$

if $I_{ii} \geq \sum_j |I_{ij}|$, $j \neq i$, $i=1, \dots, n$, then (1) is realizable.

PROOF. Choose $2n$ points on each of n oriented circles S_i , $i=1, \dots, n$. These points divide each S_i into $2n$ branches. Denote the $2n$ branches of S_i by $b_{i1}, o_{i1}, b_{i2}, o_{i2}, \dots, b_{in}, o_{in}$ with the b 's separated by the o 's. To each o is assigned the resistance zero. We shall next identify b_{ij} and b_{ji} , $i \neq j$.¹ In the resulting figure there will be a circle that is the natural topological image of S_i . We shall say that this circle is S_i .

¹ I.e., we bring the two end-points of b_{ij} into coincidence with the end-points of b_{ji} and replace these two branches by a single branch. To do this it may be necessary to replace any figure by its homeomorph.

The identification of b_{ij} and b_{ji} can be performed in two ways. If $I_{ij} > 0$, the identification is made so that S_i and S_j agree in orientation along their common branch; if $I_{ij} < 0$, S_i and S_j have opposite orientations along their common branch; and if $I_{ij} = 0$, we choose either identification. To the common branch is assigned the resistance $|I_{ij}|$.

To b_{ii} , $i = 1, \dots, n$, is assigned the resistance $I_{ii} - \sum_{j \neq i} |I_{ij}|$.

In the resulting network the S_i , $i = 1, \dots, n$, are circuits whose intersection matrix is (1).

THEOREM 2. *If matrix (1) is realizable, and if a matrix (1a) is obtained from (1) by increasing a diagonal term I_{ii} to I'_{ii} , then (1a) is realizable.*

PROOF. Consider a realization of (1) and let C_i be the circuit whose total resistance is I_{ii} . We break C_i at one of its vertices A and insert two branches b_i and b between the two ends A_1 and A_2 of the broken circuit. Any other branch that had been connected at A may now be connected at either A_1 or A_2 . To the branch b_i is assigned the resistance $I'_{ii} - I_{ii}$, while to b is assigned the resistance zero. The new network gives a realization of (1a).

The above theorems show that if all except the diagonal elements of a symmetric matrix are given, then it is possible to assign the diagonal elements so that the resulting matrix is realizable. Also there are minimal sets of diagonal elements that can be assigned. But there is not a unique minimal set of diagonal elements as we now show by example.

We wish to find the smallest a , b , and c so that

$$(2) \quad \begin{vmatrix} a & 1 & 1 \\ 1 & b & -1 \\ 1 & -1 & c \end{vmatrix}$$

is realizable. We can take $a = 1$. To do so we must have the second and third circuits S_2 and S_3 intersecting the first circuit S_1 along the same branch and both S_2 and S_3 agreeing in orientation with S_1 along this branch. Since $I_{23} = -1$, we must have S_2 and S_3 intersecting along another branch. This branch can have resistance 2 with S_2 and S_3 oppositely oriented here. This gives a realization of (2) with $a = 1$, $b = 3$, and $c = 3$.

Also we can see that if $a < 2$, then one other of the diagonal terms must be greater than 2. Hence if $a = b = c = 2$ makes (2) realizable, then this set of diagonal elements is a minimal set (in the sense that one element can not be decreased without increasing another). But

by Theorem 1 the values $a=b=c=2$ make (2) realizable. Hence there is not a unique minimal set of values for a , b , and c .

THEOREM 3. *Given I_{ij} , $i \neq j$, of (1) (but not the diagonal elements), let a be fixed; then we can take $I_{aa} = \max |I_{aj}|$, $j \neq a$, and find values for I_{ii} , $i \neq a$, so that (1) is realizable.*

The proof is similar to the proofs of the preceding theorems.

PURDUE UNIVERSITY

ON THE SPHERICAL SURFACE OF SMALLEST RADIUS ENCLOSING A BOUNDED SUBSET OF n -DIMEN- SIONAL EUCLIDEAN SPACE¹

L. M. BLUMENTHAL AND G. E. WAHLIN

1. **Introduction.** An $(n-1)$ -dimensional spherical surface $S_{n-1,r}$ is the "surface" of an n -dimensional sphere of radius r in E_n , the n -dimensional euclidean space. A given spherical surface encloses M , a subset of E_n , provided M is contained in the sphere with this surface, while M is enclosable by a given $S_{n-1,r}$ whenever M is a subset of a sphere whose surface is congruent with $S_{n-1,r}$. The purpose of this article is to show (1) *if M is any bounded subset of E_n (containing more than a single point) there exists a unique $S_{n-1,r}$ of smallest radius r enclosing M* and (2) *if d is the diameter of M , then the radius of the unique smallest $S_{n-1,r}$ enclosing M satisfies the relation $r \leq [n/2(n+1)]^{1/2} \cdot d$.*

In a proof that abounds with algebraic difficulties, H. W. E. Jung established these results in his dissertation (1901) for the case of finite point sets and indicated their extension to infinite sets at the end of his long paper.² Returning to the subject eight years later, Jung attempted a geometric proof for the case of n points in a plane, but succeeded in obtaining in this later article only necessary conditions on the smallest circle enclosing a plane (finite) set, since his procedure yields the smallest circle only in case one is assured of the existence of such a circle.³ Though this fact can readily be supplied, the geometric considerations used by Jung are not easily extended to finite

¹ Presented to the Society, February 22, 1941.

² H. W. E. Jung, *Ueber die kleinste Kugel, die eine räumliche Figur einschliesst*, Journal für die reine und angewandte Mathematik, vol. 123 (1901), pp. 241-257.

³ H. W. E. Jung, *Ueber den kleinsten Kreis, der eine ebene Figur einschliesst*, *ibid.*, vol. 137 (1909), pp. 310-313.

point sets of n -dimensional space, while for infinite sets (even of the plane) some of the argument is not valid.⁴

It seems worthwhile, therefore, to present a simple proof of the interesting assertions (1) and (2) which avoids the analytical complexities featuring Jung's demonstration and more recent proofs (see §4), deals directly with the general case of subsets of E_n in an elementary manner, and embraces in one argument both finite and infinite sets.

2. Some lemmas. Of the three lemmas established in this section, the first one is of interest apart from the application given it in this paper.

LEMMA 1. *If each set of $n+1$ points of a subset M of E_n is enclosable by $S_{n-1,r}$ of given radius r , then M is itself enclosable by this $S_{n-1,r}$.*

PROOF. Consider the family of spheres with centers at points of M and radius r . Since each set of $n+1$ points of M is enclosable by $S_{n-1,r}$ it is clear that each $n+1$ of these spheres have a point in common. Thus each $n+1$ of a family of convex bodies in E_n have a common point, and it follows from a theorem of Helly that there is a point p common to all the members of the family.⁵ Each point of M has, then, a distance from p not exceeding r , and hence M is a subset of the sphere with radius r and center p . Thus, the surface $S_{n-1,r}$ of this sphere encloses M .

This lemma permits the reduction of the problem to a finite one concerning $n+1$ points.

LEMMA 2. *Let $P = (p_1, p_2, \dots, p_{n+1})$ be a set of $n+1$ points of E_n with diameter $d > 0$. There exists a positive number r such that P is enclosable by $S_{n-1,r}$ and not enclosable by any S_{n-1,r^*} with $r^* < r$.*

The proof follows readily from the fact that bounded subsets of the E_n are compact.

We establish now some properties of an $S_{n-1,r}$ of smallest radius enclosing an independent set P of $n+1$ points, $p_1, p_2, \dots, p_n, p_{n+1}$, of E_n .

⁴ Thus, for example, the obvious property of containing at least one point of the finite set on its circumference is not necessarily carried over for infinite sets. Again, in the treatment of the plane finite case by Rademacher and Toeplitz (*Von Zahlen und Figuren*, 2d edition, 1933, pp. 83-89), it is shown that no arc of the smallest circle as large as a semicircle can be free from points of the set. This also may be invalid when infinite sets are considered.

⁵ E. Helly, *Über Mengen konvexer Körper mit gemeinschaftlichen Punkten*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 32 (1923), pp. 175-176. The use of Helly's theorem to prove Lemma 1, conjectured by one of us, occurs in C. V. Robinson's Missouri thesis.

PROPERTY 1. *The center c of $S_{n-1,r}$ is a point of the simplex whose vertices are the points of P .*

PROOF. If the contrary be assumed, a "face" of the simplex separates c from the vertex opposite this hyperplane. It is at once apparent that the $(n-1)$ -dimensional spherical surface S_{n-1,r^*} erected on the intersection of this hyperplane with $S_{n-1,r}$ encloses P and has a radius r^* less than r .⁶

PROPERTY 2. *If a point of P is not on $S_{n-1,r}$, then c lies in the face of the simplex opposite this point.*

PROOF. Assuming the contrary, let p_i be a point of P not on $S_{n-1,r}$, and select a Cartesian coordinate system so that the $(n-1)$ -dimensional hyperplane determined by the points $p_1, p_2, \dots, p_{i-1}, p_{i+1}, \dots, p_{n+1}$ has equation $x_n = 0$ and $x_n^{(j)} > 0$. It follows then from Property 1 that c_n , the n th coordinate of c , is positive. Let t be any positive number less than the smaller of the numbers $|S_{n-1,r}(p_i)|/2x_n^{(j)}$, c_n , and consider the $(n-1)$ -dimensional spherical surface S_{n-1,r^*} with equation

$$S_{n-1,r} + 2tx_n = 0.$$

It is clear that the left-hand member of this equation is negative or zero for each of the points $p_1, p_2, \dots, p_{i-1}, p_{i+1}, \dots, p_{n+1}$ since $S_{n-1,r}$ encloses these points and each of them is in the plane $x_n = 0$. But also

$$S_{n-1,r}(p_i) + 2tx_n^{(j)} < S_{n-1,r}(p_i) + |S_{n-1,r}(p_i)| = 0,$$

according to the selection of t , and hence S_{n-1,r^*} encloses all of the $n+1$ points of P . This is impossible, for since S_{n-1,r^*} is a linear combination of $S_{n-1,r}$ and the plane $x_n = 0$, it passes through the intersection of these loci, and since $0 < t < c_n$ the center $(c_1, c_2, \dots, c_{n-1}, c_n - t)$ of S_{n-1,r^*} is nearer the plane $x_n = 0$ than is the center c of $S_{n-1,r}$. Hence $r^* < r$ and the minimum property of r is contradicted.

Remark 1. It follows from Properties 1 and 2 that if c is an interior point of the simplex with vertices $p_1, p_2, \dots, p_{n+1}$ then $S_{n-1,r}$ is the surface of the sphere circumscribing the simplex.

Remark 2. The surface of the sphere circumscribing a simplex is an $(n-1)$ -dimensional spherical surface of smallest radius enclosing the

⁶ In a properly selected Cartesian coordinate system the separating hyperplane has equation $x_n = 0$, and the n th coordinates c_n and $x_n^{(n+1)}$ of c and p_{n+1} (the opposite vertex), respectively, differ in sign. S_{n-1,r^*} has equation $S_{n-1,r} + 2c_n x_n = 0$, and hence evidently encloses the $n+1$ points $p_1, p_2, \dots, p_{n+1}$, while $r^* < r$ since the center of S_{n-1,r^*} is the orthogonal projection of c on $x_n = 0$.

vertices of the simplex if and only if the circumcenter is a point of the simplex. Since we shall not use this simple criterion in establishing the general theorem, we omit its proof.

LEMMA 3. *Let P be a set of $n+1$ points of E_n , not of E_{n-1} , of diameter d . If $S_{n-1,r}$ is an $(n-1)$ -dimensional spherical surface of smallest radius r enclosing P , then $r \leq [n/2(n+1)]^{1/2} \cdot d$.*

PROOF. Let $P_1, P_2, \dots, P_{n+1}$ be vectors corresponding to the $n+1$ points $p_1, p_2, \dots, p_{n+1}$ of P , respectively, while C denotes the vector corresponding to the center c of $S_{n-1,r}$. From Property 1, c is a point of the simplex whose vertices are the points of P and hence *non-negative* constants $k_1, k_2, \dots, k_{n+1}$ exist such that

$$(1) \quad C = k_1 P_1 + k_2 P_2 + \dots + k_{n+1} P_{n+1},$$

with $k_1 + k_2 + \dots + k_{n+1} = 1$, and we may suppose the labeling of the $n+1$ points so that (2) $k_{n+1} \geq k_i$ ($i=1, 2, \dots, n+1$). Then k_{n+1} is surely positive and hence c does not lie in the face of the simplex opposite p_{n+1} . It follows from Property 2 that p_{n+1} is on $S_{n-1,r}$.

Translating the origin of coordinates to p_{n+1} does not change the constants $k_1, k_2, \dots, k_{n+1}$, and we have the scalar product $P_i \cdot (2C - P_i)$ of the vectors P_i and $2C - P_i$ equal to zero for each index i such that $k_i > 0$, for if k_i is positive then p_i is on $S_{n-1,r}$. Hence, for each such index i , $2P_i \cdot C = P_i \cdot P_i$, but the equality $2k_i P_i \cdot C = k_i (P_i \cdot P_i)$ evidently holds for every $i=1, 2, \dots, n+1$, since k_i is positive or zero. Summing for $i=1, 2, \dots, n$, we get

$$\sum_{i=1}^n (k_i P_i) \cdot C = \frac{1}{2} \sum_{i=1}^n k_i (P_i \cdot P_i) = \frac{1}{2} \sum_{i=1}^n k_i d_i^2,$$

where d_i denotes the length of P_i . Using (1) and recalling that the diameter of P is d , we conclude (since $P_{n+1} = 0$) that $C \cdot C \leq \frac{1}{2} d^2 \sum_{i=1}^n k_i$, from which it follows at once (since $\sum_{i=1}^n k_i = 1 - k_{n+1}$ and using (2)) that $r^2 \leq [n/2(n+1)] \cdot d^2$; that is, $r \leq [n/2(n+1)]^{1/2} \cdot d$, and the lemma is proved.⁷

Apart from the uniqueness (which is proved in the general theorem that follows), the Lemmas 2 and 3 prove our theorem for simplices in E_n , while Properties 1 and 2 and the accompanying remarks give important characteristics of a smallest enclosing $S_{n-1,r}$. Lemma 1 presents the means of extending the proof to any bounded subset of E_n .

⁷ The treatment by vectors was suggested by C. Herring as a more elegant presentation than the one originally obtained.

3. **The theorem.** We are now in a position to prove quite easily the principal theorem.

THEOREM. *Let d be the diameter of the bounded set M (containing more than a single point) of the n -dimensional euclidean space E_n . Then (1) there exists a unique smallest spherical surface $S_{n-1,r}$ enclosing M and (2) $r \leq [n/2(n+1)]^{1/2} \cdot d$.*

PROOF. The theorem is obviously valid for $n=1$ (the desired $S_{0,r}$ evidently consists of the endpoints of the closure of M). We make the inductive hypothesis of its validity for every positive integer k less than n .

Case 1. M is a subset of E_k , $1 \leq k < n$. Then by the inductive hypothesis there exists a unique smallest $S_{k-1,r}$ enclosing M and $r \leq [k/2(k+1)]^{1/2} \cdot d < [n/2(n+1)]^{1/2} \cdot d$ since $k < n$. It is clear that the $S_{n-1,r}$ which is the surface of the n -dimensional sphere whose radius is r and whose center coincides with that of the k -dimensional sphere with surface $S_{k-1,r}$ satisfies the requirements of the theorem.

Case 2. M is not a subset of E_k , $k < n$. Then the set $\{P\}$ of all sets of $n+1$ points of M is not empty, and by Lemma 2 there is a smallest $S_{n-1,r(P)}$ enclosing each P of $\{P\}$. Define $r = \text{l.u.b.}_{P \in \{P\}} r(P)$. Since $0 < r(P) < d$, $P \in \{P\}$, r is a positive (finite) number.

ASSERTION. M is enclosable by $S_{n-1,r}$ and by no spherical surface of smaller radius.

First, since $r \geq r(P)$, $P \subset M$, it follows that each set of $n+1$ points of M is enclosable by $S_{n-1,r}$ and hence, by Lemma 1, M is itself enclosable by $S_{n-1,r}$. Second, an assumption that M is enclosable by S_{n-1,r^*} , $r^* < r$, leads at once (according to the definition of r) to the existence of a subset P of $n+1$ points of M with $r(P) > r^*$; that is, the smallest spherical surface enclosing this subset P has a radius exceeding r^* . Hence this subset (and consequently M) is not enclosable by an S_{n-1,r^*} .

Let $S_{n-1,r}(p)$ denote an $(n-1)$ -dimensional spherical surface of smallest radius r enclosing M with center p . If, now, $S_{n-1,r}(q)$ is another such spherical surface, then M is contained in the common part of the two n -dimensional spheres of radius r and centers p and q , and consequently is part of an n -dimensional sphere of radius $r^* < r$. Then M is enclosable by S_{n-1,r^*} , $r^* < r$, which is impossible. Hence the center as well as the radius of $S_{n-1,r}$ is fixed and the uniqueness is established.

Finally, $r(P) \leq [n/2(n+1)]^{1/2} \cdot d$ for each P of $\{P\}$. For if P is in E_k , $k < n$, this follows from Case 1, while if P consists of $n+1$

independent points then the result follows from Lemma 3. Hence $r = \text{l.u.b. } P_6(P)r(P) \leq [n/2(n+1)]^{1/2} \cdot d$, and the proof of the theorem is complete.

If $p_1, p_2, \dots, p_{n+1}$ are the vertices of an equilateral simplex of edge d then the unique $(n-1)$ -dimensional spherical surface of smallest radius enclosing the points is, according to Remark 2, the surface of the circumscribed sphere. Since for this sphere $r = [n/2(n+1)]^{1/2} \cdot d$, the inequality proved for r in the theorem cannot be sharpened.

4. **Some historical remarks.** Though Jung does not mention the fact in his dissertation, some aspects of the problem he dealt with had been considered long before. Thus Sylvester, in a paper on approximate valuation of surd forms, asserted that the essential preliminary question to be resolved was that of cutting off by a plane the smallest possible segment of a sphere that should contain the whole of a given (finite) set of points lying on the surface.⁸ He stated that some years earlier he had proposed the problem of drawing the smallest circle enclosing a given finite set of points in the plane "without any suspicion of its having a practical application," and that "by a singular coincidence, Professor Peirce, of Cambridge University (*sic*), U.S., has studied this question and obtained a complete solution" which was applicable to the analogous problem on the sphere.

Unaware of either Sylvester's or Peirce's connection with the problem, G. Chrystal (1884) gave a geometrical algorithm for drawing a smallest circle enclosing a given finite set of points in the plane.⁹ No mention is made in any of this earlier work of a relationship between the diameter of the set and the radius of the smallest circle containing it.

In 1914 M. Bricard, apparently unaware of either of Jung's two earlier papers, proved that every closed plane curve of elongation (diameter) d is contained in a circle of radius $d/3^{1/2}$ and every closed surface of diameter d is contained in a sphere of radius $(3/8)^{1/2}d$.¹⁰ In a brief note H. Lebesgue remarked that Bricard's methods might be applied to any plane set E of diameter d by considering E as a subset of the set E^* which is *saturated with respect to having diameter*

⁸ J. J. Sylvester, *On Poncelet's approximate valuation of surd forms*, Philosophical Magazine, vol. 20 (1860), pp. 203-222; see also *Collected works*, vol. 2, pp. 118, 183.

⁹ G. Chrystal, *On the problem to construct the minimum circle enclosing n given points in a plane*, Proceedings of the Edinburgh Mathematical Society, vol. 3 (1884-1885), pp. 30-33.

¹⁰ M. Bricard, *Théorèmes sur les courbes et les surfaces fermées*, Nouvelles Annales de Mathématiques, (4), vol. 14 (1914), pp. 19-25.

d .¹¹ Such sets E^* are convex domains bounded by curves that Euler called *orbiform* (that is, curves of constant breadth).

A kind of dual of Jung's theorem was proved by W. Blaschke (1914).¹² If E is a plane convex set, the breadth of E is defined as the minimum distance of two parallel supporting lines of E . Blaschke showed that *the greatest circle which is contained in every plane convex set of breadth 1 has diameter $2/3$* , and established analogous results for higher dimensions. It was pointed out by J. v. Sz. Nagy that Blaschke's methods could be applied to prove Jung's theorem.¹³ In 1917 K. Reinhardt sought to extend to the E_n the geometrical arguments used by Jung in 1909 for subsets of E_2 .¹⁴ The demonstration is tedious and not entirely non-intuitive. An analogue of Jung's theorem in more general spaces was proved in 1938 by F. Bohnenblust.¹⁵

Finally, we note that in 1905 E. Landau applied Jung's theorem in the plane to sharpen an inequality in the theory of analytic functions due to F. Schottky.¹⁶

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¹¹ H. Lebesgue, *Sur les courbes orbiformes, à propos d'une note récente de M. R. Bricard*, Bulletin de la Société Mathématique de France, Comptes Rendus des Séances, 1914, pp. 45-46 (abstract).

¹² W. Blaschke, *Über den grössten Kreis in einer konvexen Punktmenge*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 369-374.

¹³ J. v. Sz. Nagy, *Über einen Satz von H. Jung*, *ibid.*, vol. 24 (1915), pp. 390-392.

¹⁴ K. Reinhardt, *Über die kleinste Kugel, die um jede Punktmenge vom Durchmesser Eins gelegt werden kann*, *ibid.*, vol. 25 (1917), pp. 157-163.

¹⁵ F. Bohnenblust, *Convex regions and projections in Minkowski spaces*, Annals of Mathematics, (2), vol. 39 (1938), pp. 301-308.

¹⁶ E. Landau, *Über einige Ungleichheitsbeziehungen in der Theorie der analytischen Funktionen*, Archiv der Mathematik und Physik, (3), vol. 11 (1905), pp. 31-36.

THE ACYCLIC ELEMENTS OF A PEANO SPACE

A. D. WALLACE

We suppose throughout that S is a Peano space. The notion of a cyclic element of such a space was introduced by G. T. Whyburn¹ and A -sets were introduced independently by W. L. Ayres² and Whyburn. Our present purpose is to consider a class of sets which may be regarded as the duals of cyclic elements.

Let $Q(S)$ denote the set of all cut-points and end-points of S .³ Then each component of $Q(S)$ will be called an *acyclic element* of S . If p and q are two points of S then we write $p \sim q$ to mean that *no point separates p from q in S* . A set will be termed *acyclic* if it contains no simple closed curve, *cyclic* if each pair of points is on a simple closed curve of the set.

(i) *Each cyclic element [acyclic element] is a cyclic [acyclic] A -set.*

PROOF. We prove only the second statement. Let F be an acyclic element, $x_n \in F$ and suppose that $x_n \rightarrow x$. If x is not in F then since $F + x$ is connected we must have $x \in S - Q(S)$, from the definition of F as a component of $Q(S)$. Consequently x is a point of a cyclic element E of S . If $(F + x) \cdot E$ contained only the point x then it would follow that F was contained in a single complementary domain R of E and thus $x = F(R) = \bar{R} - R$. Consequently x would be a cut-point, which is obviously impossible. We conclude that $(F + x) \cdot E$ is a nondegenerate connected set. It is clear that no point of this set is an end-point and hence the set contains uncountably many cut-points. But no cyclic element contains more than a countable number of cut-points. We conclude that $x \in F$ and hence F is closed. If F is not an A -set there is an arc pxq which meets F in the set $p + q$. It is manifest that no point separates p from q in S since F is connected. That is, $p \sim q$ and hence $p + q \subset E$, a true cyclic element. By the argument given

¹ See Kuratowski and Whyburn, *Fundamenta Mathematicae*, vol. 16 (1930), p. 305. By a cyclic element we understand a *nondegenerate* set such that any two points lie on a simple closed curve and which is maximal relative to this property. This is not the definition given by Kuratowski and Whyburn but is equivalent (cf. G. T. Whyburn, this Bulletin, vol. 38 (1931), p. 429 and references given there) and is in fact Whyburn's original definition. It seems more natural in the present setting.

² See W. L. Ayres, this Bulletin, vol. 46 (1940), p. 794, for references. An A -set is a *closed* arc set.

³ The terms *cut-point* and *end-point* without qualification refer to the space S . The proofs of all the statements concerning cyclic elements will be found in Kuratowski and Whyburn. We assume considerable familiarity with this fundamental work.

above E would contain uncountably many cut-points, a contradiction. Thus F is an A -set. Since each A -set is a Peano space it is easy to see that F contains no simple closed curve.

(ii) *In order that a set X of S be a cyclic element [acyclic element] it is necessary and sufficient that it be a connected set which is maximal relative to the property: If p and q are points of X then $p \sim q$ [$p \text{ non} \sim q$].*

PROOF. We prove only the second statement. Let F be an acyclic element of S , $p, q \in F$, and let t be the unique arc from p to q in F . Since F is a tree there is a point x on t which separates p from q in F . But since F is an A -set we know that x separates p from q in S so that we have $p \text{ non} \sim q$. Let F' be a connected set which contains F and which further contains at least one point which does not belong to F . Because F' is connected it contains a point p of $S - Q(S)$ and hence a point of E , a cyclic element. It is easy to see that if F' and E have in common only the point p then p must be a cut-point, which is impossible since $p \in S - Q(S)$. Hence $F' \cdot E$ contains another point q . But since $p, q \in E$ we have $p \sim q$ and by assumption $p \text{ non} \sim q$ because $p, q \in F'$. Thus $F = F'$. The condition of the theorem is thus necessary. It is not hard to see that the condition is sufficient.

(iii) *No point of a cyclic element E is an end-point or a cut-point of E while every point of an acyclic element F is either an end-point or a cut-point of F .*

We use the term *element* indiscriminately for cyclic or acyclic element.

(iv) *The elements of S form a null sequence in the sense that for any positive number δ there are only finitely many whose diameters exceed δ .*

PROOF. The result is known for cyclic elements. If there existed an infinite sequence of acyclic elements with diameters bounded from zero we could find an infinite subsequence with the same property and which converged to a nondegenerate continuum, that is, we have $F_n \rightarrow X$, $\delta(X) \geq d > 0$. If $a, b \in X$ and $a \text{ non} \sim b$, then we may write $S = M + N$, where $a \in M$, $b \in N$, $M \cdot N = p$, a cut-point. For all large n it follows that F_n meets both M and N and hence $p \in F_n$, contrary to the fact that $F_i \cdot F_j = 0$ for $i \neq j$. We conclude that $X \subseteq E$, a cyclic element. It is easy to see that we can find two points $u, v \in X - \sum F_n$, since each acyclic element meets E in at most one point and the set X is uncountable. For n sufficiently large we can find arcs uu' and vv' which are disjoint and such that $F_n \cdot uu' = u'$, $F_n \cdot vv' = v'$. Since F_n is an A -set we can find an arc $u'v'$ in F_n . Further we can find an arc uv

in E . The set $uv + u'v' + uu' + vv'$ contains a simple closed curve J which has two points in F_n . Since we may suppose that J meets E in $u + v$ we must have $J \subset E$ and thus F_n meets E in more than one point.

(v) *A Peano space is the sum of its elements and no two have more than one point in common. No two acyclic elements meet.*

(vi) *The decomposition of S into acyclic elements and points of $S - Q(S)$ is upper semi-continuous. Thus the associated monotone transformation $TS = S'$ of S onto the hyper-space S' is topological on each cyclic element of S and the acyclic elements of S' are all degenerate.*

The results stated do not exhaust the list of dual properties but we have tried to give the most striking. It appears improbable that the idea could be extended to non-locally connected spaces using the definitions given by Kelley,⁴ Moore,⁵ or Whyburn;⁶ or to the generalizations due to Hall⁷ and Youngs.⁸

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⁴ J. L. Kelley, Proceedings of the National Academy of Sciences, vol. 26 (1940), p. 192.

⁵ R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932.

⁶ G. T. Whyburn, American Journal of Mathematics, vol. 56 (1934), p. 133.

⁷ D. W. Hall, Transactions of this Society, vol. 47 (1940), p. 305.

⁸ J. W. T. Youngs, American Journal of Mathematics, vol. 52 (1940), p. 449.

ON CERTAIN BASIC SERIES

JOHN A. DAUM

1. Introduction. The identity

$$(1) \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \left\{ \frac{1}{1-q} + \frac{1}{1-q^2} + \cdots + \frac{1}{1-q^n} \right\} = \sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^n}$$

was deduced from arithmetical considerations by E. T. Bell. About five years ago, W. N. Bailey¹ proved the relation

$$(2) \sum_{n=0}^{\infty} \frac{(1-q)(1-q^2) \cdots (1-q^n)}{(1-z)(1-qz) \cdots (1-q^n z)} \cdot \frac{z^{n+1}}{1-q^{n+1}} = \sum_{n=0}^{\infty} \frac{q^n z}{(1-q^n z)^2},$$

from which he obtained (1) by differentiating with respect to z and then putting $z=q$. A short time later Hall² gave an alternate proof of (2) by simply specializing the parameters in a relation between basic series.³

Since Bell's identity (1) leads to interesting arithmetical results, it would be desirable to obtain as many more similar relations as possible. Hall's paper, as well as Bailey's, seems to indicate a connection with certain formulae in basic series. The identity which Hall employed to obtain Bailey's relation is the basic analogue of but one of many identities involving the series ${}_3F_2$ which were proved originally by Thomae⁴ and later classified by Whipple.⁵

The object of this paper is to investigate the possibility of systematically obtaining new results similar to Bailey's by employing the complete set of basic analogues of Thomae's two-term transformations.

2. The basic analogues of Thomae's relations. We first establish the complete set of basic analogues of Thomae's relations, using a notation analogous to that used by Whipple.

Let $I = (a, b, c, d, e, f)$ be a matrix whose elements form an arbitrary permutation of the integers 0, 1, 2, 3, 4, 5 and let $r_a, r_b, r_c, r_d, r_e, r_f$ be six numbers such that their product is unity. Define parameters A and B by

¹ W. N. Bailey, *An algebraic identity*, Journal of the London Mathematical Society, vol. 11 (1936), p. 156.

² N. A. Hall, *An algebraic identity*, *ibid.*, vol. 11 (1936), p. 276.

³ W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tract, no. 32, p. 65.

⁴ J. Thomae, Journal für die reine und angewandte Mathematik, vol. 87 (1879), p. 26.

⁵ F. J. Whipple, Proceedings of the London Mathematical Society, vol. 23 (1925), p. 104.

$$A_{uvw} = r_u r_v r_w q^{1/2}, \quad |q| < 1; \quad B_{uv} = r_u q / r_v;$$

and functions Φp and Φn by

$$\Phi p(a; b, c) = \frac{1}{\Omega(A_{def})\Omega(B_{ba})\Omega(B_{ca})} {}_3\Phi_2 \left[\begin{matrix} A_{dbc}, A_{ebc}, A_{fbc}; A_{def} \\ B_{ba}, B_{ca} \end{matrix} \right],$$

$$\Phi n(a; b, c) = \frac{1}{\Omega(A_{abo})\Omega(B_{ab})\Omega(B_{ao})} {}_3\Phi_2 \left[\begin{matrix} A_{ade}, A_{adf}, A_{aef}; A_{abo} \\ B_{ab}, B_{ao} \end{matrix} \right],$$

where⁶

$${}_3\Phi_2 \left[\begin{matrix} \alpha, \beta, \gamma; z \\ \delta, \epsilon \end{matrix} \right] = \frac{\Omega(\delta)\Omega(\epsilon)}{\Omega(\alpha)\Omega(\beta)\Omega(\gamma)} \sum_{n=0}^{\infty} \frac{\Omega(\alpha q^n)\Omega(\beta q^n)\Omega(\gamma q^n)}{\Omega(q^{n+1})\Omega(\delta q^n)\Omega(\epsilon q^n)} z^n, \quad |z| < 1,$$

and⁷

$$\Omega(x) = \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - xq^n}.$$

Considering all possible matrices I , it is possible to form 60 Φp 's and 60 Φn 's.

There are 90 fundamental relations between certain pairs of the functions Φp . The proof of these relations, which can all be written in the form

$$(3) \quad \Phi p(a; b, c) = \Phi p(a; d, e),$$

is almost identical with that of the original case as given by Bailey,⁸ subject to certain analogous conditions. The only essential difference is in the replacement of the gamma-function throughout by the modified Heine omega-function.

The 90 identities (3) are all equivalent in the sense that any one of these identities may be obtained directly from any other by proper determination of the r 's. The corresponding relations involving the functions Φn are all equivalent to those involving the Φp 's—one set being obtainable from the other by replacing the r 's by their reciprocals.

As an example of the use of relations (3), consider the identity

$$\Phi p(0; 4, 5) = \Phi p(0; 2, 3),$$

where

⁶ W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tract, no. 32, p. 65.

⁷ N. A. Hall, loc. cit.

⁸ W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tract, no. 32, p. 14.

$$A_{145} = \alpha, \quad A_{245} = \beta, \quad A_{345} = \gamma, \quad B_{40} = \delta, \quad B_{80} = \epsilon, \quad A_{128} = \delta\epsilon/\alpha\beta\gamma.$$

We obtain the identity

$${}_3\Phi_2 \left[\begin{matrix} \alpha, \beta, \gamma; \delta\epsilon/\alpha\beta\gamma \\ \delta, \epsilon \end{matrix} \right] = \frac{\Omega(\delta)\Omega(\epsilon)\Omega(\delta\epsilon/\alpha\beta\gamma)}{\Omega(\alpha)\Omega(\delta\epsilon/\alpha\beta)\Omega(\delta\epsilon/\alpha\gamma)} {}_3\Phi_2 \left[\begin{matrix} \delta/\alpha, \epsilon/\alpha, \delta\epsilon/\alpha\beta\gamma; \alpha \\ \delta\epsilon/\alpha\beta, \delta\epsilon/\alpha\gamma \end{matrix} \right],$$

which was used by Hall to obtain Bailey's identity (2).

For a fixed value of a (3) represents 15 identities between certain of the functions $\Phi p(a)$. For $a=0$, these identities are

$$\begin{aligned} \Phi p(0; 4, 5) &= \Phi p(0; 2, 3) & \Phi p(0; 1, 5) &= \Phi p(0; 2, 3) \\ &= \Phi p(0; 1, 3) & &= \Phi p(0; 2, 4) \\ &= \Phi p(0; 1, 2), & &= \Phi p(0; 3, 4), \\ \\ \Phi p(0; 3, 5) &= \Phi p(0; 1, 2) & \Phi p(0; 1, 4) &= \Phi p(0; 2, 3) \\ (4) \quad &= \Phi p(0; 1, 4) & \Phi p(0; 2, 4) &= \Phi p(0; 1, 3) \\ &= \Phi p(0; 2, 4), & \Phi p(0; 3, 4) &= \Phi p(0; 1, 2), \\ \\ \Phi p(0; 2, 5) &= \Phi p(0; 1, 3) \\ &= \Phi p(0; 1, 4) \\ &= \Phi p(0; 3, 4). \end{aligned}$$

From this, it is clear that we have a second set of identities of the form

$$(5) \quad \Phi p(a; b, c) = \Phi p(a; b, d)$$

subject to certain conditions obtained from those applied in proving (3). Relation (5) represents 180 equivalent identities. Typical of results obtained are

$$(6) \quad {}_3\Phi_2 \left[\begin{matrix} \alpha, \beta, \gamma; \delta\epsilon/\alpha\beta\gamma \\ \delta, \epsilon \end{matrix} \right] = \frac{\Omega(\delta)\Omega(\epsilon)\Omega(\delta\epsilon/\alpha\beta\gamma)}{\Omega(\delta\epsilon/\alpha\beta)\Omega(\epsilon)\Omega(\delta/\gamma)} {}_3\Phi_2 \left[\begin{matrix} \epsilon/\beta, \epsilon/\alpha, \gamma; \delta/\gamma \\ \delta\epsilon/\alpha\beta, \epsilon \end{matrix} \right]$$

and

$$(7) \quad {}_3\Phi_2 \left[\begin{matrix} \delta\epsilon/\alpha\beta\gamma, \delta/\alpha, \epsilon/\alpha; \alpha \\ \delta\epsilon/\alpha\gamma, \delta\epsilon/\alpha\beta \end{matrix} \right] = \frac{\Omega(\delta\epsilon/\alpha\gamma)\Omega(\delta\epsilon/\alpha\beta)\Omega(\alpha)}{\Omega(\delta\epsilon/\beta\gamma)\Omega(\delta\epsilon/\alpha\beta)\Omega(\beta)} {}_3\Phi_2 \left[\begin{matrix} \delta\epsilon/\alpha\beta\gamma; \delta/\beta, \epsilon/\beta; \beta \\ \delta\epsilon/\beta\gamma, \delta\epsilon/\alpha\beta \end{matrix} \right].$$

It is possible to obtain Bailey's identity from either (6) or (7).

3. **Conclusion.** This classification of the basic analogues of Thomae's two-term relations indicates, then, that out of the entire set of identities, only two are essentially distinct. Any further results can be obtained from this source only by specialization of the parameters in Hall's identity or in relation (6) or (7).

Analogous statements can be made concerning the corresponding relations between the generalized hypergeometric functions ${}_3F_2$ as given by Whipple.

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A NOTE ON FINITE REGULAR RINGS

JOHN DYER-BENNET

The purpose of this note is to give a classification of all finite regular rings. One result of the classification is the extension of a part of Wedderburn's classic theory of the structure of linear associative algebras to systems which are not algebras. I wish to thank Professor Garrett Birkhoff for several suggestions in the preparation of the note.

DEFINITION. *A ring R is said to be regular if for every element a of R there exists an element x in R such that*

$$(1) \qquad aka = a.^1$$

LEMMA 1. *A regular ring has radical (0).²*

PROOF. Since the radical of a ring is a nilpotent ideal, any element a in it has the property that a product containing a as a factor a sufficient number of times is zero. Since by definition there exists an x such that $axax \cdots axa = a$ no matter how many terms there are in the product, zero must be the only member of the radical.

¹ Cf. von Neumann, *Lectures on Continuous Geometries*, Part 2, Princeton, 1937, pp. 7-21. Also *On regular rings*, Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 707-713. von Neumann postulates the existence of a unit, but in our case this follows from the other assumptions.

² Cf. von Neumann, loc. cit.

From Lemma 1 it follows that a regular ring which satisfies the finite descending chain condition for ideals is semi-simple, and hence contains a unit.

DEFINITION. *The characteristic of a ring is the least common multiple of the additive orders of its elements, if this is finite, otherwise ∞ .*

If the ring has a unit this is the additive order of the unit, so that this definition agrees with the usual one for a field.

LEMMA 2. *The characteristic of a regular ring with a unit is square-free, if it is finite.*

PROOF. Suppose the characteristic c is not square-free. Let

$$c = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

where we may assume that $e_1 > 1$. Let $a = c/p_1$ where we are considering a as an element of the ring. Then $a \neq 0$. Further, for any x in the ring

$$axa = aax = 0 \neq a.$$

It follows that the ring is not regular.

THEOREM 1. *A finite regular ring is the direct sum of algebras.*

PROOF. By the theory of commutative groups the additive group of the ring is the direct sum of subgroups each of which consists of all the elements whose order is a fixed prime p .³ Since the order of a product of two elements of order p is also of order p —unless the product is zero—these subgroups are also subrings. Next, let a and b be two elements of additive orders p and q respectively, $p \neq q$. Then

$$pab = (pa)b = 0 \cdot b = 0$$

and

$$qab = a(qb) = a \cdot 0 = 0,$$

where pc stands for a sum of p summands c . Thus the order of ab is less than or equal to p and to q , and divides both. Hence $ab = 0$. Since the product of any two elements from different subrings is zero, the ring is the direct sum of these subrings. Each of these subrings is of prime characteristic, and hence contains in its center a prime field of p elements. Over this field it is clearly a linear associative algebra. Finally, each of these is clearly semi-simple.

³ Cf. Speiser, *Theorie der Gruppen von endlicher Ordnung*, 3d edition, p. 49.

THEOREM 2. *A finite regular ring is the direct sum of full matrix algebras over finite fields, and the summands are unique, apart from their order.*

PROOF. By Theorem 1 it is the direct sum of semi-simple algebras. By a theorem due to Wedderburn each of these is a direct sum of simple algebras, and these are unique, apart from their order.⁴ But a simple algebra is a total matrix algebra over a division algebra.⁵ Finally, every finite division algebra is commutative, and hence a field.⁶

The proof of Theorem 2 above seems to lend itself to generalization, but it is not the simplest proof that can be given. By relying even more heavily on results in the literature the theorem can be proved as follows.

von Neumann has shown that a regular ring with a unit is irreducible if and only if its center is a field.⁷ Hence a finite regular ring is the direct sum of (finite regular) rings whose centers are fields. The terms in this direct sum are uniquely determined.⁸ This reduces the problem to that of classifying the finite regular rings whose centers are fields. Such a ring is a normal simple linear associative algebra over its center,⁹ therefore a total matrix algebra over its center.¹⁰

There is an immediate corollary of the results above. Since a regular ring is irreducible if and only if its center is a field, there follows the theorem (due to Dedekind¹¹) that a commutative regular ring satisfying the descending chain condition is the direct sum of fields.

We can also get another proof of what is almost Lemma 2, namely that a semi-simple ring has a square-free characteristic if the characteristic is finite. For the direct sum of two fields of characteristic p is of characteristic p , while the direct sum of two fields of characteristics p and q respectively has characteristic pq . This result is interesting for the following reason. The regularity condition (1) above is a natural weakening of the condition that inverses should exist,¹² and so a regular ring might be thought of as somewhere between an arbi-

⁴ Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, p. 39.

⁵ Cf. Albert, loc. cit., p. 39.

⁶ Cf. Albert, loc. cit., p. 62.

⁷ *Continuous Geometries*, Part 2, p. 21.

⁸ van der Waerden, *Moderne Algebra*, vol. 2, p. 162.

⁹ Albert, loc. cit., pp. 6, 37.

¹⁰ Albert, loc. cit., p. 62.

¹¹ Cf. van der Waerden, loc. cit., p. 163.

¹² Cf. von Neumann, *Continuous Geometries*, Part 2, p. 21.

trary ring and a field. Similarly the result that the characteristic is square-free is a natural weakening of the result that the characteristic of a finite field is prime.

We complete the note by giving a combinatory formula for the number of non-isomorphic regular rings of a given finite order. The formula is unsatisfactory in that it involves finding all the partitions of an integer. It does not seem likely, however, that this can be avoided.

The algebra of n^2 matrices over a field of p^k elements has $p^{n^2 k}$ elements. Hence there are as many total matrix algebras with p^e elements as there are squares that divide e . Let

$$e = p_1^{e_1} \cdots p_r^{e_r}.$$

Then there are clearly

$$\prod_{i=1}^r \{ [e_i/2] + 1 \}$$

such algebras, where $[a]$ is the greatest integer in a . These are all different, since the center of each is isomorphic with the field over which it is taken.

The number of non-isomorphic regular rings, each of which is the direct sum of s simple rings of orders p^{e_i} respectively is the product of the numbers for each of the e_i . The number of order p^k is therefore the sum of these products for all possible partitions of k , $k = \sum_{i=1}^s e_i$.

Using these results and Theorem 2 we see that the number of non-isomorphic rings of order $n = \prod_{i=1}^r p_i^{k_i}$ is

$$\prod_{i=1}^r \sum_{e_1 + \cdots + e_s = k_i} \prod_{i=1}^s \prod_{j=1}^{r_i} \{ [e_{i,j}/2] + 1 \}$$

where the summation is taken over all partitions of k_i , and

$$e_i = \prod_{j=1}^{r_i} p_i^{e_{i,j}}.$$

OSCULATING QUADRICS OF RULED SURFACES IN RECIPROCAL RECTILINEAR CONGRUENCES

M. L. MACQUEEN

1. **Introduction.** Let x be a general point of an analytic non-ruled surface S referred to its asymptotic net in ordinary projective space. By a line l_1 at the point x we mean any line through the point x and not lying in the tangent plane of the surface at the point x . Dually, a line l_2 is any line in the tangent plane of the surface at the point x but not passing through the point x . The lines l_1, l_2 are called reciprocal lines if they are reciprocal polar lines with respect to the quadric of Lie at the point x . In this case, when the point x varies over the surface S , the lines l_1, l_2 generate two rectilinear congruences Γ_1, Γ_2 which are said to be reciprocal with respect to the surface. If, however, the point x moves along the u -curve, the locus of the line l_1 is a ruled surface $R_1^{(u)}$ of the congruence Γ_1 . The osculating quadric along a generator l_1 of the ruled surface $R_1^{(u)}$ is the limit of the quadric determined by the line l_1 through the point x and the lines l_1 through two neighboring points P_1, P_2 on the u -curve as each of these points independently approaches the point x along the u -curve. The quadric thus defined will be denoted by $Q_1^{(u)}$. A second quadric $Q_1^{(v)}$ is determined by three consecutive lines l_1 at points of the v -curve through the point x . Moreover, there are two quadrics, denoted by $Q_2^{(u)}$ and $Q_2^{(v)}$, which are associated with two ruled surfaces of the reciprocal congruence Γ_2 and which can be defined similarly. This note will study the projective differential geometry of the quadrics thus defined.

2. **Analytic basis.** Let the surface S under consideration be an analytic non-ruled surface whose parametric vector equation, referred to asymptotic parameters u, v , is

$$(1) \quad x = x(u, v).$$

The four coordinates x of a variable point x on the surface satisfy two partial differential equations which can be reduced, by a suitably chosen transformation of proportionality factor, to Fubini's canonical form

$$(2) \quad x_{uu} = px + \theta_u x_u + \beta x_v, \quad x_{vv} = qx + \gamma x_u + \theta_v x_v, \quad \theta = \log \beta\gamma,$$

in which the subscripts indicate partial differentiation. The coefficients of these equations are functions of u, v and satisfy three integrability conditions which need not be written here.

Two lines l_1, l_2 are reciprocal lines if the line l_1 joins the point x and the point y defined¹ by

$$(3) \quad y = -ax_u - bx_v + x_{uv}$$

and the line l_2 joins the points ρ, σ defined by placing

$$(4) \quad \rho = x_u - bx, \quad \sigma = x_v - ax,$$

where a, b are functions of u, v . As u, v vary, the lines l_1, l_2 generate two rectilinear congruences Γ_1, Γ_2 which are reciprocal with respect to the surface.

The curves corresponding to the developables of the congruence Γ_1 are called the Γ_1 -curves of the congruence, and those corresponding to the developables of the congruence Γ_2 the Γ_2 -curves of the congruence. The differential equation of the Γ_1 -curves is

$$(5) \quad (F - 2a\beta + \beta\psi)du^2 - (b_v - a_u)dudv - (G - 2b\gamma + \gamma\phi)dv^2 = 0,$$

where F, G are defined by the formulas

$$F = p - b_u + b\theta_u - b^2 + a\beta, \quad G = q - a_v + a\theta_v - a^2 + b\gamma,$$

and ϕ, ψ by

$$\phi = (\log \beta\gamma^2)_u, \quad \psi = (\log \beta^2\gamma)_v.$$

If k_1, k_2 are the roots of the equation

$$(6) \quad k^2 + (A + B)k + AB - (F - 2a\beta + \beta\psi)(G - 2b\gamma + \gamma\phi) = 0,$$

where the functions A, B are defined by

$$A = -a_u - ab + \beta\gamma + \theta_{uv}, \quad B = -b_v - ab + \beta\gamma + \theta_{uv},$$

the corresponding points

$$y + k_i x, \quad i = 1, 2,$$

are the focal points of the line l_1 . Furthermore, the differential equation of the Γ_2 -curves is

$$(7) \quad Fdu^2 - (b_v - a_u)dudv - Gdv^2 = 0.$$

If τ_1, τ_2 are the roots of the equation

$$(8) \quad F + (b_v - a_u)\tau - G\tau^2 = 0,$$

the corresponding points

$$\rho + \tau_i \sigma, \quad i = 1, 2,$$

¹ In this section we employ the notation used by E. P. Lane in Chapter III of his book *Projective Differential Geometry of Curves and Surfaces*, Chicago, 1932.

are the focal points of the line l_2 . It will be assumed that the coefficients of du^2 and dv^2 in equations (5), (7) are all nonzero. In this case the Γ_1 -curves and the Γ_2 -curves of two reciprocal congruences form conjugate nets if, and only if, $b_v - a_u = 0$.

3. **Osculating quadrics of ruled surfaces of the congruence Γ_1 .** Any point z , except the point y , on the line l_1 at the point x is given by the equation

$$(9) \quad z = x + \omega y, \quad \omega \text{ scalar.}$$

As the point x varies along the u -curve, the line l_1 generates a ruled surface $R_1^{(u)}$. Equation (9) is the parametric vector equation of this ruled surface, u, ω being the independent parameters. The asymptotic curves on $R_1^{(u)}$ consist of the lines l_1 and the integral curves of the differential equation

$$(10) \quad L_1 du + 2M_1 d\omega = 0,$$

where L_1, M_1 are determinants of the fourth order defined by

$$L_1 = (z_{uu}, z, z_u, z_\omega), \quad M_1 = (z_{u\omega}, z, z_u, z_\omega).$$

Differentiating equation (9) and using equations (2), (3) to calculate the values of L_1, M_1 , we find that equation (10) can be written in the form

$$(11) \quad \frac{d\omega}{du} = - \frac{\beta + C\omega + D\omega^2}{2(F - 2a\beta + \beta\psi)},$$

where we have placed

$$\begin{aligned} C &= F_u - 2(a\beta)_u + (\beta\psi)_u + 2\beta A, \\ D &= \beta A^2 - a(F - 2a\beta + \beta\psi)^2 + A(F - 2a\beta + \beta\psi)_u \\ &\quad - (F - 2a\beta + \beta\psi)[p_v + \beta q - ap + A(b - \theta_u) + A_u]. \end{aligned}$$

Any point X , except the point z , on the tangent at the point z of the curved asymptotic on $R_1^{(u)}$ is defined by placing

$$(12) \quad X = \lambda z + dz/du, \quad \lambda \text{ scalar.}$$

If we use the tetrahedron x, ρ, σ, γ as a local tetrahedron of reference with a unit point chosen so that a point

$$x_1x + x_2\rho + x_3\sigma + x_4\gamma$$

has local coordinates proportional to $x_1, \dots, x_4$, we find that the local coordinates of the point X are given by

$$\begin{aligned}
 x_1 &= b + \lambda + [a(F - 2a\beta + \beta\psi) + bA + p_v + \beta q - ap]\omega, \\
 x_2 &= 1 + A\omega, \\
 x_3 &= (F - 2a\beta + \beta\psi)\omega, \\
 x_4 &= \omega\lambda - (b - \theta_u)\omega - \frac{\beta + C\omega + D\omega^2}{2(F - 2a\beta + \beta\psi)}.
 \end{aligned}
 \tag{13}$$

Homogeneous elimination of ω, λ from these equations gives the algebraic equation of the quadric $Q_1^{(u)}$, referred to the tetrahedron x, ρ, σ, y , namely

$$\begin{aligned}
 &\beta(F - 2a\beta + \beta\psi)x_2^2 + Hx_3^2 + 2(F - 2a\beta + \beta\psi)^2x_2x_4 \\
 &- 2A(F - 2a\beta + \beta\psi)x_3x_4 \\
 &- 2(F - 2a\beta + \beta\psi)x_1x_3 + 2Px_2x_3 = 0,
 \end{aligned}
 \tag{14}$$

where the coefficients H, P are defined by

$$\begin{aligned}
 H &= a(F - 2a\beta + \beta\psi) - 3A(b - \theta_u) - A_u + p_v + \beta q - ap, \\
 P &= (2b - \theta_u)(F - 2a\beta + \beta\psi) + \frac{1}{2}(F - 2a\beta + \beta\psi)_u.
 \end{aligned}
 \tag{15}$$

The equation of the quadric $Q_1^{(v)}$ can be written by interchanging u and v and making the appropriate symmetrical interchanges of the other symbols. The result is

$$\begin{aligned}
 &Kx_2^2 + \gamma(G - 2b\gamma + \gamma\phi)x_3^2 + 2(G - 2b\gamma + \gamma\phi)^2x_2x_4 \\
 &- 2B(G - 2b\gamma + \gamma\phi)x_3x_4 \\
 &- 2(G - 2b\gamma + \gamma\phi)x_1x_2 + 2Qx_2x_3 = 0,
 \end{aligned}
 \tag{16}$$

where the coefficients K, Q are given by

$$\begin{aligned}
 K &= b(G - 2b\gamma + \gamma\phi) - 3B(a - \theta_v) - B_v + q_v + \gamma p - bq, \\
 Q &= (2a - \theta_v)(G - 2b\gamma + \gamma\phi) + \frac{1}{2}(G - 2b\gamma + \gamma\phi)_v.
 \end{aligned}
 \tag{17}$$

Some properties of the quadrics $Q_1^{(u)}, Q_1^{(v)}$ will now be deduced. In the first place, the tangent plane, $x_4=0$, intersects each of the quadrics in a conic. The conic of intersection of the tangent plane and the quadric $Q_1^{(u)}$ touches the u -tangent at the point x and intersects the v -tangent in the point whose local coordinates are

$$\left(\frac{1}{2}H, 0, F - 2a\beta + \beta\psi, 0\right).
 \tag{18}$$

Similarly, the quadric $Q_1^{(v)}$ is intersected by the tangent plane in a conic which is tangent to the v -tangent at the point x and intersects the u -tangent in the point

$$(19) \quad (\tfrac{1}{2}K, G - 2b\gamma + \gamma\phi, 0, 0).$$

The face $x_3=0$ of the tetrahedron of reference intersects the quadric $Q_1^{(u)}$ in the line l_1 and in the line whose equations are

$$(20) \quad \beta x_2 + 2(F - 2a\beta + \beta\psi)x_4 = 0, \quad x_3 = 0.$$

Moreover, the face $x_3=0$ cuts the quadric $Q_1^{(u)}$ in the line l_1 and in the line which joins the point (18) to the point on the line l_1 with local coordinates

$$(21) \quad (-A, 0, 0, 1).$$

Similarly, the face $x_2=0$ cuts the quadric $Q_1^{(v)}$ in the line l_1 and in the line

$$(22) \quad \gamma x_3 + 2(G - 2b\gamma + \gamma\phi)x_4 = 0, \quad x_2 = 0.$$

The face $x_2=0$ cuts the quadric $Q_1^{(v)}$ in the line l_1 and in the line which passes through the point (19) and meets the line l_1 in the point

$$(23) \quad (-B, 0, 0, 1).$$

The points (21), (23) are found to coincide if, and only if, $a_u = b_v$. Thus we reach the following conclusion:

The Γ_1 -curves and the Γ_2 -curves of two reciprocal rectilinear congruences form conjugate nets on the surface if, and only if, the points (21), (23) coincide.

It is well known that two nonsingular quadric surfaces having one, and only one, generator in common intersect elsewhere in a twisted cubic. Elimination of x_1 between equations (14), (16) gives the cubic cone projecting the curve of intersection of the two quadrics from the point x . This cone has the line l_1 for a double line, the equations of the nodal tangent planes along the line l_1 being given by

$$(24) \quad (F - 2a\beta + \beta\psi)x_2^2 - (b_v - a_u)x_2x_3 - (G - 2b\gamma + \gamma\phi)x_3^2 = 0.$$

A glance at equation (5) suffices to substantiate the following statement:

The nodal tangent planes along the double line l_1 of the cone projecting the curve of intersection of the quadrics $Q_1^{(u)}$, $Q_1^{(v)}$ from the point x are the planes which intersect the tangent plane of the surface at the point x in the tangents of the Γ_1 -curves.

Eliminating x_2 from equations (14), (16), we obtain the equation of the cone which projects the curve of intersection of the quadrics

$Q_1^{(u)}, Q_1^{(v)}$ from the vertex ρ of the tetrahedron of reference. This projecting cone is found to be a composite quartic cone, one component being the face $x_3=0$ of the tetrahedron of reference. The other component is a cubic cone which is intersected by the face $x_3=0$ in a plane cubic curve. Placing $x_3=0$ in the equation of this curve, we find the intersections of the curve with the line l_1 . It is now easy to verify the conclusion:

The quadrics $Q_1^{(u)}, Q_1^{(v)}$ intersect in the line l_1 and in a twisted cubic which crosses the line l_1 in its two focal points.

4. **Osculating quadrics of ruled surfaces of the congruence Γ_2 .** The equations of the quadrics $Q_2^{(u)}, Q_2^{(v)}$ can be found without difficulty by applying the method of the preceding section. The details of the calculation need not be reproduced here, but the required *equation of the quadric $Q_2^{(u)}$, referred to the tetrahedron x, ρ, σ, γ* , is found to be

$$(25) \quad \beta x_1^2 - 2Fx_1x_3 + 2F^2x_3x_4 + 2(ab - a_u)Fx_3x_4 + 2Sx_1x_4 + Lx_4^2 = 0,$$

where the functions S, L are defined by

$$(26) \quad \begin{aligned} S &= \frac{1}{2}F_u - F\theta_u + 2bF - \beta(ab - a_u), \\ L &= \beta FG + FF_v - F(\theta_u - 2b)(b_v - a_u) + F(b_v - a_u)_u \\ &\quad - 2S(ab - a_u) - \beta(ab - a_u)^2. \end{aligned}$$

The equation of the quadric $Q_2^{(v)}$ is

$$(27) \quad \gamma x_1^2 - 2Gx_1x_3 + 2G^2x_3x_4 + 2(ab - b_v)Gx_3x_4 + 2Tx_1x_4 + Mx_4^2 = 0$$

where

$$(28) \quad \begin{aligned} T &= \frac{1}{2}G_v - G\theta_v + 2aG - \gamma(ab - b_v), \\ M &= \gamma FG + GG_u - G(\theta_v - 2a)(a_u - b_v) + G(a_u - b_v)_v \\ &\quad - 2T(ab - b_v) - \gamma(ab - b_v)^2. \end{aligned}$$

The quadric $Q_2^{(u)}$ is intersected by the tangent plane in the line l_2 and also in the line

$$(29) \quad \beta x_1 - 2Fx_3 = 0, \quad x_4 = 0.$$

The face $x_3=0$ cuts the quadric $Q_2^{(u)}$ in a conic which is tangent to the u -tangent at the point ρ and which intersects the edge $x_1=x_3=0$ in the point with local coordinates

$$(30) \quad (0, L, 0, -2F^2).$$

The face $x_1=0$ cuts the quadric $Q_2^{(w)}$ in the line l_2 and in the line which joins the point

$$(31) \quad (0, a_u - ab, F, 0)$$

on the line l_2 to the point (30).

Similarly, the tangent plane intersects the quadric $Q_2^{(v)}$ in the line l_2 and in the line

$$(32) \quad \gamma x_1 - 2Gx_2 = 0, \quad x_4 = 0.$$

The plane $x_2=0$ cuts this quadric in a conic which is tangent to the v -tangent at the point σ and which intersects the edge $x_1=x_2=0$ in the point

$$(33) \quad (0, 0, M, -2G^2).$$

The face $x_1=0$ intersects the quadric $Q_2^{(v)}$ in the line l_2 and in the line which joins the point

$$(34) \quad (0, G, b_v - ab, 0)$$

on the line l_2 to the point (33). The following conclusion is immediate.

If the points (31), (34) coincide respectively with the points σ , ρ , the Γ_1 -curves and the Γ_2 -curves form conjugate nets.

Elimination of x_2 from equations (25), (27) yields the equation of the cubic cone projecting from the point ρ the curve of intersection of the quadrics $Q_2^{(w)}$, $Q_2^{(v)}$. The line l_2 is a double line of this cone, the nodal tangent planes along the line l_2 being given by

$$(35) \quad x_1^2 - (b_v - a_u)x_1x_4 - [FG + (ab - b_v)(ab - a_u)]x_4^2 = 0.$$

It is now easy to verify the conclusion:

The two nodal tangent planes along the double line l_2 of the cone projecting the curve of intersection of the quadrics $Q_2^{(w)}$, $Q_2^{(v)}$ from the point ρ intersect the line l_1 in two points which separate the points x , y harmonically if, and only if, the Γ_1 -curves and the Γ_2 -curves form conjugate nets.

Finally, simple calculations suffice to demonstrate the following theorem:

The quadrics $Q_2^{(w)}$, $Q_2^{(v)}$ intersect in the line l_2 and in a twisted cubic which cuts the line l_2 in its two focal points.

5. A special case. The theory of the preceding sections will now be specialized by considering a particular covariant pair of reciprocal

lines associated with the point x of the surface. It is known that the line l_1 is the projective normal and the line l_2 is the reciprocal projective normal in case $a=b=0$ in equations (3), (4). Placing $a=b=0$ in equations (14), (16), one easily shows that the equations of the two quadrics $Q_1^{(u)}$, $Q_1^{(v)}$, which we shall call *the projective normal quadrics*, are respectively

$$(36) \quad \begin{aligned} & \beta\pi x_2^2 + (p_v + \beta q + 3\theta_u k - k_u)x_3^2 + 2\pi^2 x_2 x_4 + (\pi_u - 2\pi\theta_u)x_2 x_3 \\ & \quad - 2\pi k x_3 x_4 - 2\pi x_1 x_3 = 0, \\ & \gamma\chi x_3^2 + (q_u + \gamma p + 3\theta_v k - k_v)x_2^2 + 2\chi^2 x_3 x_4 + (\chi_v - 2\chi\theta_v)x_2 x_3 \\ & \quad - 2\chi k x_3 x_4 - 2\chi x_1 x_2 = 0, \end{aligned}$$

where π, χ, k are defined by the formulas

$$\pi = p + \beta\psi, \quad \chi = q + \gamma\phi, \quad k = \beta\gamma + \theta_{uv}.$$

Moreover, by placing $a=b=0$ in equations (25), (27), we obtain *the two reciprocal projective normal quadrics* $Q_2^{(u)}$, $Q_2^{(v)}$, whose equations are respectively

$$(37) \quad \begin{aligned} & \beta x_1^2 + p(p_v + \beta q)x_4^2 + (p_u - 2p\theta_u)x_1 x_4 - 2p x_1 x_3 + 2p^2 x_2 x_4 = 0, \\ & \gamma x_1^2 + q(q_u + \gamma p)x_4^2 + (q_v - 2q\theta_v)x_1 x_4 - 2q x_1 x_2 + 2q^2 x_3 x_4 = 0. \end{aligned}$$

INDECOMPOSABLE CONNEXES¹

PAUL M. SWINGLE

DEFINITION. A connected set M is an indecomposable connexe if and only if, for every two connected subsets H and K of M such that $M=H+K$, either H and M or K and M have the same closure.²

Any connected subset N of an indecomposable continuum W , which is dense in W , such as any set of composants of W or W itself, is an indecomposable connexe, as is also a widely connected set.³

EXAMPLE A.⁴ Let, in a euclidean plane, U be the points of a square, Q , plus its interior. Let U_i ($i=1, 2, 3, \dots$) be a set of mutually exclusive arcs each contained in U and having one and only one point, an end point, common with Q . Let the U_i 's be taken so that every plane region of U is joined to every linear region of Q by at least one U_i . Let $M=U-(U_1+U_2+\dots)$. Then M is connected⁵ and such that, if H and K are connected and their sum is M , either H and M or K and M have the same closure. Hence M is an indecomposable connexe.

EXAMPLE B. Let, in a euclidean plane, U be the points of a triangle plus its interior, one vertex of which is the point a . Let U_i ($i=1, 2, 3, \dots$) be a set of arcs, mutually exclusive, except for having the common end point a , and whose sum is dense in U . Let further the U_i 's be taken so that each two plane regions of U are joined by at least one U_i . Let $M=U-(U_1+U_2+\dots)$. It can be shown without difficulty that M is an indecomposable connexe.

¹ Presented to the Society November 23, 1940.

² See S. Eilenberg, *Topology du plan*, Fundamenta Mathematicae, vol. 26, p. 81, for a definition of an indecomposable connected space. This definition is seen to be equivalent to the above for the types of spaces considered in these two papers.

³ For definition and example see P. M. Swingle, *Two types of connected sets*, this Bulletin, vol. 37 (1931), pp. 254-258.

⁴ E. W. Miller communicated this interesting example to me by letter in 1937 calling attention to its relation to a widely connected set. The method of construction is somewhat similar to the well known boring process used to obtain a plane indecomposable continuum. See K. Yoneyama, *Theory of continuous sets of points*, Tôhoku Mathematical Journal, vol. 12 (1917), p. 60. That either H and M or K and M have the same closure is seen above by supposing that neither H nor K is dense in M , from which it readily follows that H and K can each have at most one point common with Q itself.

⁵ E. W. Miller, *Some theorems on continua*, this Bulletin, vol. 46 (1940), p. 153, Theorem 3.

It is proposed to give here a generalization of some of the well known theorems on indecomposable continua⁶ by means of indecomposable connexes and the following definitions. The imbedding space will be one satisfying R. L. Moore's Axioms 0 and 1.⁷

DEFINITIONS. A connected subset K of a connected set M will be called a proper connexe subclosure of M if and only if M and K do not have the same closure. A connected set M is an irreducible connexe closure between two points a and b if and only if M contains $a+b$ and there does not exist a proper connexe subclosure of M containing $a+b$. A connected set M is an irreducible joining connexe closure between a and b if and only if there exists a subset N of M such that both N and $N+a+b$ are connected and, for all such N 's, M and N have the same closure.

Both a continuum and a connected set, irreducible between two points, are irreducible connexe closures between these two points. Also a widely connected set is an irreducible connexe closure between any two of its points. It is seen readily that if M is an irreducible connexe closure between a and b , then M is an irreducible joining connexe closure between a and b .

EXAMPLE C. In a euclidean plane let B be a biconnected set with dispersion point a and containing the point b distinct from a . Let W be an arc-wise connected set such that (a) if x and y are any two points of W then $W+a$ contains arcs ax and ay such that one of these contains the other, (b) for each x there exists but one arc ax , (c) the closure of $W+a-ax$ contains B , and (d) the product of ax and the closure of B is a . Then $M=W+B-a-b$ is an irreducible joining connexe closure from a to b , since each connected subset N of M , such that $N+a+b$ is connected, contains W . However $M+a+b$ is not an irreducible connexe closure from a to b , since $M+a+b$ contains B , which contains $a+b$, and B and M do not have the same closure.

DEFINITIONS. A connected subset K of a connected set M is a connexe of condensation of M if and only if every point of K is a limit point of $M-K$. If M is connected a composant of M is a set of points K_p , which consists of a point p , of the closure of M but not necessarily of M , and of all points x of M such that there exists a proper connexe subclosure containing $p+x$ and contained in M excepting perhaps for p .

⁶ Brouwer, *Zur Analysis situs*, Mathematische Annalen, vol. 68 (1910), p. 426, gave the first example and definition of indecomposable continuum. For theorems on these sets see Z. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fundamenta Mathematicae, vol. 1, p. 215.

⁷ *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, 1932.

And K_p is a composant of M if and only if p is also contained in M , i.e., if K_p is a component of $M+$ but is contained entirely in M .

In a widely connected set M each composant of M consists of but one point, and each composant of $M+$ may consist of but one point. Hence it is not true that if K is such a composant of M every point of M is a limit point of K , which is however a useful theorem on indecomposable continua.⁸

THEOREM 107'. *Every composant of $M+$, where M is connected and its closure is compact, is the sum of a countable number of proper connexe subclosures, each contained in M except perhaps for one point of the closure of M .*

PROOF. Let a be a point of the closure of M and let K denote the composant of $M+$ consisting of a and all points x of M such that $M+a$ is not an irreducible connexe closure from a to x . Then there exists⁹ a countable set G of domains such that if q is any point of the closure of M and D is any domain containing q there exists a domain of G , containing q , and contained wholly in D . For each domain R of G , which does not contain a , let M_R denote the maximal connected subset, containing a , of $(M+a) \cdot (S-R)$, S being the imbedding space. Let H denote the collection of all sets M_R and let T denote the sum of all these proper connexe subclosures of $M+a$ which are elements of H . The set H is countable. If q is a point of $M-T$ then $M+a$ is an irreducible connexe closure from a to q . For if there exists a proper connexe subclosure N of $M+a$, containing $a+q$, there exists a domain g of G such that the product of the closures of g and N is vacuous, where g contains a point of M . Thus N would have been contained in an M_R above and so N , and thus q , would be contained in T . Therefore K is T . Hence K is the sum of a countable number of proper connexe subclosures as the theorem states.

COROLLARY 107'. *If M is connected and its closure is compact, then every composant of M is the sum of a countable number of proper connexe subclosures of M .*

LEMMA A. *If M is an indecomposable connexe and N is a proper connexe subclosure of M , then $M - M \cdot \bar{N}$ is connected.¹⁰*

⁸ R. L. Moore, loc. cit., Theorem 106, p. 75. Below, the theorems are numbered to correspond to similar theorems on indecomposable continua, given by Moore, pp. 75-78. It is to be noted the methods of proof are somewhat similar.

⁹ R. L. Moore, loc. cit., Theorem 19, p. 14.

¹⁰ By $\bar{N}$ is meant the closure of N .

PROOF. Suppose $M - M \cdot \bar{N}$ is the sum of the two mutually separated sets H and K . Then M is the sum of two proper connexe¹¹ subclosures $H + \bar{N} \cdot M$ and $K + \bar{N} \cdot M$ and so M is not indecomposable.

LEMMA A'. If M is an indecomposable connexe and N is a proper connexe subclosure of M , then $M - N$ is connected.

PROOF. By Lemma A $M - M \cdot \bar{N}$ is connected. Also $\bar{N} \cdot M$ is connected since N is. As M is the sum of these two sets and $M \cdot \bar{N}$ is a proper connexe subclosure, $M - M \cdot \bar{N}$ cannot be proper.

Suppose $M - N$ is the sum of the mutually separate sets U and V . But $M - N$ contains the connected set $M - M \cdot \bar{N}$ and so either U or V contains it also. Say U does. Then M and U must have the same closure. But then points of V are limit points of U which is a contradiction. Hence $M - N$ is connected.

THEOREM A. If M is an indecomposable connexe and W a connected subset of M such that M and W have the same closure, then W is an indecomposable connexe.

PROOF. Let $N = M - W$ and suppose $W = H + K$, H and K proper connexe subclosures of W . As N is contained in $\bar{W} = \bar{H} + \bar{K}$, let $\bar{H} \cdot N = H'$ and $\bar{K} \cdot N = K'$. Thus $H + H'$ and $K + K'$ are connected sets.¹² But as $\bar{H}$ contains the closure of $H + H'$ and $\bar{K}$ the closure of $K + K'$, $M = W + N$ is the sum of these two proper connexe subclosures and so M is not indecomposable.

COROLLARY A. If M is an indecomposable connexe and N is both a proper connexe subclosure and a connexe of condensation of M , then $M - N$ is an indecomposable connexe.

PROOF. By Lemma A $M - N$ is connected and by definition of connexe of condensation M and $M - N$ have the same closure. Thus the corollary follows from Theorem A.

COROLLARY A'. If $M + f$ is an indecomposable connexe, M connected and f finite, then M is an indecomposable connexe.

Theorem A and its corollaries treat the case where an indecomposable connexe is given and the subtraction of points gives an indecomposable connexe. This suggests the following addition problem: Let M be an indecomposable connexe and p a point of $\bar{M} - M$. Is $M + p$ an indecomposable connexe? This problem is left unsolved here.

¹¹ R. L. Moore, loc. cit., Theorem 47, p. 33.

¹² R. L. Moore, Theorem 27, p. 17.

THEOREM 108'. *Let M be connected. Then in order that M be an indecomposable connexe it is necessary and sufficient that every proper connexe subclosure of M be a connexe of condensation of M .*

PROOF. The condition is sufficient. For suppose M is not indecomposable. Then M is the sum of two proper connexe subclosures H and K . Thus there exists a point q of H which is not a limit point of K . But K contains $M-H$. Thus q is not a limit point of $M-H$ and so H is not a connexe of condensation of M .

The condition is necessary. For suppose N is a proper connexe subclosure of M but that not every point of N is a limit point of $M-N$. By Lemma A' $M-N$ is connected but the closures of M and $M-N$ are not the same. Hence M is the sum of two proper connexe subclosures N and $M-N$ which is a contradiction.

THEOREM 108''. *Let M be connected. Then in order that M be an indecomposable connexe it is necessary and sufficient that the closure of every proper connexe subclosure of M be a continuum of condensation of the closure of M .*

PROOF. The condition is sufficient. For suppose H and K are as in the proof above and that q is a point of $\bar{H}$ which is not a limit point of K . As $\bar{M} = \bar{H} + \bar{K}$ and $q \cdot \bar{K} = 0$ q is not a limit point of $\bar{M} - \bar{H}$ contained in $\bar{K}$. Thus $\bar{H}$ is not a continuum of condensation of $\bar{M}$.

The condition is necessary. As M is indecomposable, by Lemma A, $M - M \cdot \bar{N}$ is connected, where N is a proper connexe subclosure of M . Hence M is the sum of the two connected sets $M - M \cdot \bar{N}$ and $M \cdot \bar{N}$, the latter being a proper connexe subclosure of M . Hence $M - M \cdot \bar{N}$ is not proper and so every point of $M \cdot \bar{N}$, and so of N , is a limit point of $M - M \cdot \bar{N}$. Thus every point of $\bar{N}$ is a limit point of $M - M \cdot \bar{N} = (M + \bar{N}) - \bar{N}$. Therefore every point of $\bar{N}$ is a limit point of $\bar{M} - \bar{N}$ and so $\bar{N}$ is a continuum of condensation of $\bar{M}$.

Let B be a composant of an indecomposable continuum K , where $K-B$ contains an arc A . Let c and d be two points of A such that $A - c - d = A' + A'' + A'''$, where A' , A'' , and A''' are mutually separated sets, but $A' + c + A''$ and $A'' + d + A'''$ are connected. Let $M = B + A' + A'' + A'''$. Then M is an indecomposable connexe. The composant of $M +$ containing c is $A' + c + A''$ and the one containing d is $A'' + d + A'''$. Thus two composants of $M +$ are not necessarily mutually exclusive.

THEOREM 109'. *If M is an indecomposable connexe, whose closure is compact, then no two composants of M have a point in common.*

PROOF. For each point p of M let M_p denote the set of points x such that M is not an irreducible connexe closure from p to x . If b is a point of M_a then $M_b = M_a$. For suppose not and that x is any point of M_a and y is of M_b . Then there exist proper connexe subclosures N_{ax} , N_{by} , N_{ab} of M . Suppose $\overline{N_{ab}} + \overline{N_{ax}} = \overline{M}$. But N_{ab} and N_{ax} are continua of condensation of $\overline{M}$ by Theorem 108''. This is a contradiction.¹³ Therefore $N_{ab} + N_{ax}$ is a proper connexe subclosure of M as is similarly $(N_{ab} + N_{ax}) + N_{by}$. Hence $N_{ab} + N_{ax} + N_{by}$ is contained in both M_a and in M_b and so $M_a = M_b$. Hence if two composants have a point in common they are the same composant.

Since a composant of an indecomposable continuum is itself an indecomposable connexe it is not true that an indecomposable connexe contains uncountably many composants. A composant of $M+$ however may consist of a single point. Thus we have the following theorem.

THEOREM 110'. *If M is an indecomposable connexe whose closure is compact and, for every point p of $\overline{M} - M$, $M+p$ is an indecomposable connexe, then there exist an uncountable number of composants of $M+$.*

PROOF. Suppose there exist but a countable number of composants of $M+$. Then by Theorem 107' M is contained in a countable number of proper connexe subclosures of $\overline{M}$. Say these are the elements of the set (N) . An N of (N) contains at most one point p of $\overline{M} - M$ and by hypothesis $M+p$ is an indecomposable connexe. Hence by Theorem 108'' $\overline{N}$ is a continuum of condensation of $\overline{M+p} = \overline{M}$. But $\overline{M}$ is the sum of the $\overline{N}$'s of (N) , since M is the sum of the N 's. As this is a contradiction¹⁴ the theorem is true.

THEOREM 111'. *If M is connected and its closure is compact then in order that M be an indecomposable connexe it is necessary and sufficient that there exist three distinct points such that M is an irreducible joining connexe closure between any two of them.*

PROOF. The condition is sufficient. For if M is the sum of the connexes H and K , one of these has at least two of the three points as limit points and so it and M have the same closure.

The condition is necessary. For if M contains three points x , y , and z such that each of these is in a different composant, M is an irreducible connexe closure between any two of these points. Consider the case where M contains only the one composant T , containing a

¹³ R. L. Moore, loc. cit., Theorem 15, p. 11.

¹⁴ R. L. Moore, loc. cit., Theorem 15, p. 11.

point x . Then by Corollary 107' M is the sum of the elements of a countable class (N) , each element of which is a proper connexe subclosure. Then by Theorem 108'' every $\bar{N}$ of (N) is a continuum of condensation of $\bar{M}$. But if $\bar{M}$ is the sum of the $\bar{N}$'s this is a contradiction.¹⁶ Hence $\bar{M} - M$ contains points y and z which are not contained in any $\bar{N}$ of (N) . Thus if the connected set H of T contains x and has z as a limit point, H and M have the same closure. Thus M is an irreducible joining connexe closure from x to z and similarly from x to y . Suppose M contains a proper connexe subclosure N' which has y and z as limit points. Because of the nature of H above, N' does not contain x . From the manner of constructing the sets N of (N) in Theorem 107', using x for the point a there, it is seen that N' is contained in an N of (N) and so does not have y or z as a limit point. Therefore M is an irreducible joining connexe closure between y and z also. In case M is the sum of two composants, y and z can be taken as above and the proof completed.¹⁸

THEOREM 112'. *If a is a point of an indecomposable connexe M whose closure is compact and K is the set of all points x such that M is an irreducible joining connexe closure from a to x , then K is dense in M .*

PROOF. Suppose that there exists a region R , containing a point of M , such that R does not contain a point of K . Let N be a maximal connected subset of $R \cdot M$. Then by Lemma A $\bar{M} - M \cdot \bar{N}$ is connected as N is a proper connexe subclosure of M . Thus $\bar{N}$ is a continuum of condensation of $\bar{M}$. Hence¹⁷ the locally compact closed set $\bar{M} \cdot \bar{R}$ is not the sum of the closures of a countable number of composants of $M \cdot R$. Hence by Theorem 107' $\bar{M} \cdot \bar{R}$ is not contained in the sum of the closures of the countable number of proper connexe subclosures

¹⁶ R. L. Moore, loc. cit., Theorem 15, p. 11.

¹⁸ The question arises whether the condition in Theorem 111' might be changed to "there exist three points x , y , and z such that $M+x+y+z$ is an irreducible connexe closure between any two of these." That three points might be taken so that $M+y$, say, is not an irreducible connexe closure between y and some point of M is seen by the following example. Let interior to the square Q , of Example A above, (V) be the set of straight line intervals joining a Cantor ternary set, on a line t , to a point y not on t . Take the U_i 's as in Example A, except that no U_i has a point common with a V of (V) . Let $B+y$ be a biconnected subset of the sum of the elements of (V) , B being totally disconnected. Let $M = U - (U_1 + U_2 + \dots) - (\text{points of the elements of } (V)) + B$. Then M is indecomposable but $M+y$ is not an irreducible connexe closure between y and a point of B . See Example C above. Whether M could be taken so that each point of $\bar{M} - M$ is as y and $M+y$ is not an irreducible connexe closure between any two points is a question.

¹⁷ R. L. Moore, loc. cit., Theorem 15, p. 11.

of the composant of M which contains a . Therefore by a proof similar to that of Theorem 111' M is an irreducible joining connexe closure between a and some point of $(\overline{M} - M) \cdot R$. Thus K is dense in M .

If T is the sum of a countable number of proper connexe subclosures of an indecomposable connexe M , since M may be a composant of an indecomposable continuum, it is readily seen that $M - T$ may be disconnected. However, by repeated use of Lemma A, Theorem 108', and Theorem A, the following theorem is seen to be true.

THEOREM 113'. *If T is the sum of a finite number of mutually exclusive proper connexe subclosures of an indecomposable connexe M , then $M - T$ is a non-vacuous indecomposable connexe.*

The two following theorems are proven in a manner similar to that used for the corresponding theorems on continua.

THEOREM 114'. *If a is a point of a decomposable connexe M , there exists a domain D containing a such that M is not an irreducible connexe closure from a to any point of D .*

THEOREM 115'. *If a and b are two distinct points, M is an irreducible connexe closure from a to b , and T is a proper connexe subclosure of M containing b , then $M - M \cdot \overline{T}$ is connected.*

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A PROBLEM IN PARTITIONS¹

MARSHALL HALL

Let m objects $x_1, \dots, x_m$ be given and from these n non-void subsets $a_1, \dots, a_n$ be formed. This partition will determine a matrix (a_{ij}) in which $a_{ij}=1$ if a_i and a_j have a non-void intersection and $a_{ij}=0$ if a_i and a_j are disjoint. Necessarily (a_{ij}) is a symmetric matrix with 1's on the main diagonal. The following question has arisen in Ore's investigation of the theory of relations: Is every matrix (a_{ij}) , $i, j=1, \dots, n$, with $a_{ii}=1$, $a_{ij}=a_{ji}=0$ or 1 the partition matrix of n objects into n non-void subsets? As will be seen presently, the answer to this question is in the negative. The reason is not that there is any inherent contradiction within certain matrices but that it is not always possible to find a partition of as few as n objects determining a given matrix.

The answer is affirmative for $n=1, 2, 3, 4$ as may be found by direct calculation, but is negative for $n \geq 5$. It is almost trivial that for $n \geq 3$, $m=(n^2-n)/2$ objects will suffice. Take $(n^2-n)/2$ objects $u_{ij}=u_{ji}$, $i \neq j$, $i, j=1, \dots, n$, and assign u_{ij} to both a_i and a_j if $a_{ij}=1$ and discard u_{ij} if $a_{ij}=0$. This will leave certain subsets a_i void for which $a_{ii}=1$, $a_{ij}=0$ if $j \neq i$, and for these we introduce new objects u_i in a_i alone. If there are one or two such i 's we have discarded at least two u_{ij} 's since $n \geq 3$. If there are $s \geq 3$ such i 's we have discarded at least $s(s-1)/2$ u_{ij} 's, namely those with both subscripts from this set. In all events we have discarded at least as many objects as we have added and we have a partition of $(n^2-n)/2$ or fewer objects into n non-void subsets corresponding to the prescribed partition matrix. But it is clear that this number $m=(n^2-n)/2$ is too high for $n > 3$ since the full number of objects is used only if every $a_{ij}=1$ and in this case a single object assigned to every subset will suffice.

THEOREM 1. *A given matrix (a_{ij}) , $i, j=1, \dots, n$, in which $a_{ii}=1$, $a_{ij}=a_{ji}=0$ or 1, is the partition matrix of a set of at most n objects for $n=1, 2, 3, 4$ and of at most $n^2/4$ (n even and $n \geq 4$) or $(n^2-1)/4$ (n odd and $n \geq 5$).*

PROOF. Evidently renumbering the subsets makes no difference in the problem. This operation corresponds to permuting both the rows and columns of the matrix, the same permutation being applied to both. Such an operation defines an equivalence on the matrices. The

¹ Presented to the Society, February 22, 1941.

non-equivalent matrices of orders 2 and 3 are

$$(1) \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Those of order 2 require 2 and 1 objects respectively and those of order 3 require 3, 2, 2, and 1 objects respectively. There are eleven classes of matrices of order four of which only two,

$$(2) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

require as many as four objects. Four classes require three, four classes require two, and one requires one.

Proof for $n=5$. If $a_{ij}=0$ for $j \neq i$, the matrix requires exactly five objects, one in each subset. If some $a_{ij}=1$ renumber to make this a_{12} , and suppress the first two rows and columns. The remaining matrix for a_3, a_4 , and a_5 will require at most three objects, and only two unless $a_{34}=a_{35}=a_{45}=0$. Now take four objects u_{12}, u_3, u_4, u_5 . Assign u_i , $i=3, 4, 5$, to a_1 and a_i if $a_{1i}=1$ and to a_2 and a_i if $a_{2i}=1$, and discard it if $a_{1i}=a_{2i}=0$. If there is an i for which $a_{1i}=a_{2i}=1$, then u_i is common to both a_1 and a_2 and we may discard u_{12} . If not assign u_{12} to both a_1 and a_2 . These four objects and those used for the matrix for a_3, a_4, a_5 realize the partition with at most six elements save in the case in which exactly one of a_{1i}, a_{2i} is 1 for $i=3, 4, 5$ and $a_{34}=a_{35}=a_{45}=0$. Now we may discard the elements placed in a_3, a_4, a_5 originally and the partition is realized using only the four elements u_{12}, u_3, u_4 , and u_5 .

Proof for $n \geq 6$. If $a_{ij}=0$ for $j \neq i$ we assign a single element x_i to each subset a_i and realize the partition with exactly n objects. If some $a_{ij}=1$, $j \neq i$, renumber so that this is a_{12} . Now suppress the first and second rows and columns. We have a partition matrix for $n-2$ subsets which, by an induction hypothesis, may be realized with at most $(n-2)^2/4$ or $[(n-2)^2-1]/4$ objects, depending on whether n is even or odd. We now use $n-1$ new objects $u_{12}, u_3, u_4, \dots, u_n$. We assign u_{12} to both a_1 and a_2 . For $i \geq 3$ if $a_{1i}=1, a_{2i}=0$ we assign u_i to a_1

and a_i . If $a_{1i}=0$, $a_{2i}=1$ we assign u_i to a_2 and a_1 . If $a_{1i}=a_{2i}=1$ we assign u_i to a_1 , a_2 , and a_i . If $a_{1i}=a_{2i}=0$ we discard u_i . This yields a realization of the prescribed partition, using at most $n-1$ more objects. Since $(n-2)^2/4+n-1=n^2/4$ and $[(n-2)^2-1]/4+n-1=(n^2-1)/4$, the truth of the theorem for $n-2$ implies the truth for n , and the truth of the theorem for $n=4, 5$ completes the proof by induction.

We note that the new objects used are never assigned to more than three subsets. For $n=5$ this is also true, and for $n=4$ an object may be assigned to all four subsets only if every a_{ij} is 1. But here four objects u_{123} , u_{124} , u_{134} , u_{234} assigned respectively to the three appropriate subsets realize the partition. Hence it is never necessary to assign an object to more than three subsets.

THEOREM 2. *If $n=2s$ is even, the symmetric matrix (a_{ij}) with $a_{ii}=1$; $a_{ij}=0$ for $j \neq i$ and $i, j=1, \dots, s$; $a_{ij}=1$ for $i=1, \dots, s$, and $j=s+1, \dots, 2s$; $a_{ij}=0$ for $j > i$ and $i, j=s+1, \dots, 2s$, is the partition matrix of $n^2/4$ but no fewer objects. If $n=2s+1$ is odd, the matrix (a_{ij}) with $a_{ii}=1$, $a_{ij}=0$ for $j > i$ and $i, j=1, \dots, s$; $a_{ij}=1$ for $i=1, \dots, s$ and $j=s+1, \dots, 2s+1$; $a_{ij}=0$ for $j > i$ and $i, j=s+1, \dots, 2s+1$, is the partition matrix of $(n^2-1)/4$ but no fewer objects. A partition matrix of order $n \geq 5$ requiring $n^2/4$ objects (n even) or $(n^2-1)/4$ objects (n odd) is equivalent to the appropriate matrix above.*

PROOF. In any partition if an object x belongs to as many as three subsets a_i, a_j, a_k , then in the partition matrix we have $a_{ij}=a_{ik}=a_{jk}=1$. But in the above matrices $a_{ij}=a_{ik}=1$ (i, j, k different) always imply $a_{jk}=0$. For if $i \leq s$ and $a_{ij}=a_{ik}=1$, then $j \geq s+1$, $k \geq s+1$ and $a_{jk}=0$; and if $i \geq s+1$, $a_{ij}=a_{ik}=1$ then $j \leq s$, $k \leq s$ and $a_{jk}=0$. Hence the matrices above must come from partitions in which no object appears in more than two subsets. Thus in these matrices each 1 above the main diagonal corresponds to at least one object in common to two subsets and these objects must all be different. Hence these are the partition matrices of $n^2/4$ or $(n^2-1)/4$ objects but no fewer.

Consider a matrix (a_{ij}) , $n \geq 5$, requiring the maximum number of objects. If some $a_{ii}=1$ and all $a_{ij}=0$, $j \neq i$, then we may adjoin a single element for a_i to a partition for the remaining $n-1$ subsets and have a partition with less than the maximum number of objects. First suppose $n \geq 6$. In the proof of Theorem 1 for $n \geq 6$ note that if $a_{1i}=a_{2i}=0$ we discard u_i and have fewer than $n-1$ new objects to adjoin. Also if $a_{1i}=a_{2i}=1$ we may discard u_{12} since u_i is an object common to both a_1 and a_2 . In these cases the partition requires fewer

than $n^2/4$ or $(n^2-1)/4$ objects. Since, by renumbering, any triple i, j, k may be taken into $1, 2, i$, it follows that in a matrix requiring the maximum number of objects exactly two of a_{ij}, a_{ik}, a_{jk} must be 1's and the third must be 0 or all three must be 0. For a nonvanishing triple renumber so that the common subscript of the two 1's is 1 and so that the first row is of the form $a_{11}=1, a_{1i}=0, i=2, \dots, t, a_{1i}=1, i=t+1, \dots, n$. Here $t \leq n-2$ since there are at least two 1's in this row. Now $a_{ij}=0$ if $j \neq i, i, j \geq t+1$, since otherwise a_{1i}, a_{1j}, a_{ij} would all be 1's. Also $a_{ij}=1$ if $i \leq t, j \geq t+1$, since $a_{1i}=0, a_{1j}=1$. Finally $a_{ij}=0$ if $j \neq i$ and $i, j \leq t$. For $a_{ni}=1, a_{nj}=1$ by the preceding argument. This completely determines the matrix which has a rectangle containing $t(n-t)$ 1's in the upper right-hand corner and also in the lower left-hand corner but 0's elsewhere apart from the main diagonal. By placing objects u_{ij} in a_i and a_j if $a_{ij}=1, j \neq i$, we may realize this partition using $t(n-t)$ objects. These objects are certainly different since none may be in three subsets. If $n=2s$ is even $t(n-t) < n^2/4$ unless $t=s$ which yields the matrix of the theorem. If $n=2s+1, t(n-t) < (n^2-1)/4$ unless $t=s$ or $s+1$. $t=s$ yields the matrix of the theorem, while $t=s+1$ yields an equivalent matrix. The permutation $(1, n)(2, n-1) \dots (s, s+1)$ of the rows and columns interchanges these two matrices.

Finally suppose $n=5$. Here $(n^2-1)/4=6$. Suppose for some triple, say $1, 2, 3, a_{12}=a_{13}=a_{23}=1$. Then as in the proof of Theorem 1 for $n=5$, to a partition of objects for a_3, a_4, a_5 we need add at most three objects. If these three subsets required only two objects, we have used at most five objects for the entire partition. If they required three objects, then $a_{34}=a_{35}=a_{45}=0$ and we may discard the object originally placed in a_3 and have a complete partition using at most five objects. Hence there is no triple i, j, k for which $a_{ij}=a_{ik}=a_{jk}=1$. Similarly if for some triple, say $1, 2, 3, a_{12}=1, a_{13}=0, a_{23}=0$ we need to add at most three objects to the partition for a_3, a_4, a_5 and if this requires three objects then either someone of the old objects may be discarded or one of the new objects is discarded. Hence if $a_{12}=1$, one of a_{13}, a_{23} is 1 and the other 0. From here the proof as for $n \geq 6$ applies.

MEAN-VALUE SURFACES

MAXWELL READE AND E. F. BECKENBACH

Introduction. The real functions

$$(1) \quad x_j = x_j(u, v), \quad j = 1, 2, 3,$$

defined and continuous in a finite simply connected domain¹ D , will be said to define a surface S . If the first partial derivatives of the functions (1) are continuous in D , and if

$$(2) \quad E(u, v) = G(u, v), \quad F(u, v) = 0$$

hold in D , where

$$E(u, v) \equiv \sum_{j=1}^3 x_{ju}^2, \quad F(u, v) \equiv \sum_{j=1}^3 x_{ju} x_{jv}, \quad G(u, v) \equiv \sum_{j=1}^3 x_{jv}^2$$

are the coefficients of the first fundamental quadratic form of S , then the surface is said to be given in isothermic representation by the functions (1) and the parameters u, v are said to be isothermic parameters; the map of D on S is conformal except where $E = G = 0$.

In a previous paper,² the authors studied the equation

$$(3) \quad \sum_{j=1}^3 \left[\int_C x_j(u, v) dz \right]^2 = 0, \quad z = u + iv,$$

where C is a circle in D ; the following necessary and sufficient condition was obtained.

THEOREM A. *If the functions (1) have continuous partial derivatives of the third order in a finite simply connected domain D , then a necessary and sufficient condition that they map D isothermically either on a surface S that lies on a sphere of finite non-null radius, such that circles are mapped on circles, or on a minimal surface S , is that (3) hold for each circle C in D .*

1. Mean-value surfaces. Let the coordinate functions (1) of a surface S be continuous in a finite simply connected domain D ; then the circular averages

¹ A domain is a non-null connected open set.

² *Generalizations to space of the Cauchy and Morera theorems*, Transactions of this Society, vol. 49 (1941), pp. 354–377; in particular, see p. 365.

$$(4) \quad x_j = A_{j,p}(u, v) \equiv \frac{1}{\pi\rho^2} \iint_{\xi^2 + \eta^2 \leq \rho^2} x_j(u + \xi, v + \eta) d\xi d\eta, \quad j = 1, 2, 3,$$

where ρ is a positive constant, will be said to define a *mean-value surface* S_ρ associated with S . We define

$$A_{j,0}(u, v) \equiv x_j(u, v), \quad j = 1, 2, 3.$$

We note that the functions (4) are defined and have continuous partial derivatives of the first order in an open set of points D_ρ which is interior to D ; since D_ρ is not necessarily a connected set, S_ρ may consist of several pieces.

THEOREM 1.1. *If the functions (1) are continuous in a simply connected domain D , then a necessary and sufficient condition that (3) hold for each circle C in D is that all mean-value surfaces S_ρ associated with the surface S , defined by the functions (1), be given in isothermic representation by (4).*

PROOF. The first partial derivatives of the functions (4) are given by the relations³

$$\begin{aligned} \frac{\partial A_{j,p}}{\partial u} &= \frac{1}{\pi\rho^2} \int_{\xi^2 + \eta^2 = \rho^2} x_j(u + \xi, v + \eta) d\eta, \\ \frac{\partial A_{j,p}}{\partial v} &= -\frac{1}{\pi\rho^2} \int_{\xi^2 + \eta^2 = \rho^2} x_j(u + \xi, v + \eta) d\xi, \quad j = 1, 2, 3, \end{aligned}$$

which are valid for points of D_ρ ; hence

$$(5) \quad \sum_{j=1}^3 \left[\int_{\xi^2 + \eta^2 = \rho^2} x_j(u + \xi, v + \eta) (d\xi + i d\eta) \right]^2 = -\pi^2 \rho^4 [E_\rho - G_\rho + 2iF_\rho],$$

where E_ρ , F_ρ and G_ρ are the coefficients of the first fundamental quadratic form of S_ρ . From (2), (3) and (5) we obtain the theorem.

From Theorems A and 1.1 we obtain the following result.

THEOREM 1.2. *If the functions (1) have continuous partial derivatives of the third order in a finite simply connected domain D , then a necessary and sufficient condition that they map D isothermically either on a surface S that lies on a sphere of finite non-null radius, such that circles are mapped on circles, or on a minimal surface S , is that all mean-value surfaces S_ρ associated with the surface S defined by the functions (1) be given in isothermic representation by (4).*

³ T. Radó, *Subharmonic Functions*, Berlin, 1937, p. 11.

2. **Mean-value surfaces and transformations of axes.** In §3 we shall make use of the following observations.

2.1. S_ρ is invariant under rigid transformations in the (x_1, x_2, x_3) -space; if

$$x_j = a_j + \sum_{k=1}^3 \lambda_{kj} x_k, \quad j = 1, 2, 3,$$

is a rigid transformation, then

$$A'_{j,\rho}(u, v) = a_j + \sum_{k=1}^3 \lambda_{kj} A_{k,\rho}(u, v), \quad j = 1, 2, 3,$$

where

$$A'_{j,\rho}(u, v) \equiv \frac{1}{\pi \rho^2} \iint_{\xi^2 + \eta^2 \leq \rho^2} x'_j(u + \xi, v + \eta) d\xi d\eta, \quad j = 1, 2, 3.$$

2.2. S_ρ is invariant under each of the reflections

$$u' = u, v' = -v,$$

and

$$u' = -u, v' = v.$$

If, for example,

$$x'_j(u', v') \equiv x_j(u', -v'), \quad j = 1, 2, 3,$$

then

$$A'_{j,\rho}(u', v') = A_{j,\rho}(u', -v'), \quad j = 1, 2, 3,$$

where

$$(6) \quad A'_{j,\rho}(u', v') \equiv \frac{1}{\pi \rho^2} \iint_{\xi^2 + \eta^2 \leq \rho^2} x'_j(u' + \xi, v' + \eta) d\xi d\eta, \quad j = 1, 2, 3.$$

2.3. Similarly, S_ρ is invariant under rigid transformations in the (u, v) -plane.

2.4. Under the transformation

$$(7) \quad u' = \alpha u, \quad v' = \alpha v, \quad \alpha > 0,$$

the mean-value surface S_ρ is transformed into the mean-value surface $S'_{\alpha\rho}$ whose coordinate functions are given by (6), where

$$x'_j(u', v') \equiv x_j\left(\frac{u'}{\alpha}, \frac{v'}{\alpha}\right), \quad j = 1, 2, 3.$$

Hence for α fixed, the *family* of mean-value surfaces $[S_p]$ associated with a given surface S is identical with the *family* of mean-value surfaces whose coordinate functions are given by (6); under the transformation (7), each member of one family is congruent to a member of the second family.

3. Conformal mean-value surfaces. Since, by Theorem 1.2, the only smooth surfaces in isothermic representation for which all associated mean-value surfaces are given in isothermic representation by (4) are (a) spherical surfaces, in representation whereby circles are mapped on circles, and (b) minimal surfaces, the question arises as to the nature of the mean-value surfaces in these two cases.

THEOREM 3.1. *If the functions (1) map a finite simply connected domain D isothermically on a minimal surface S , then each mean-value surface S_p associated with S is a minimal surface given in isothermic representation by (4) and coinciding with S for (u, v) in D_p .*

PROOF. By a theorem of Weierstrass,⁴ the functions (1) are harmonic in D ; consequently it follows from the mean-value property of harmonic functions that the functions (4) coincide with the functions (1) in the open set D_p . Hence all mean-value surfaces associated with minimal surfaces given in isothermic representation by (1) are themselves minimal surfaces given in isothermic representation by (4); the surface S_p coincides with S for (u, v) in D_p .

THEOREM 3.2. *If the functions (1) are not identically constant and if they map a finite simply connected domain D isothermically on a surface S that lies on a sphere $\mathbb{S}$ of finite non-null radius a , such that circles are mapped on circles, then each mean-value surface S_p associated with S lies on a surface of revolution T_p and is given in isothermic representation by (4). Further, for $0 < p < \infty$, T_p is not a sphere.*

PROOF. It has been pointed out, in a recently published paper,⁵ that the functions (1) may be continued isothermically to map the entire closed u, v -plane isothermically on the whole of $\mathbb{S}$; further, the functions (1) have the representation

⁴ If the functions (1) are harmonic in D , and if (2) holds in D , then the functions (1) are said to form a *triple of conjugate harmonic functions*; see E. F. Beckenbach and T. Radó, *Subharmonic functions and minimal surfaces*, Transactions of this Society, vol. 35 (1933), pp. 648-661. Then the theorem of Weierstrass may be stated as follows. *A necessary and sufficient condition that the functions (1), defined in the domain D , be the coordinate functions of a minimal surface given in isothermic representation is that they form a triple of conjugate harmonic functions*; loc. cit., p. 649.

⁵ Loc. cit., see Footnote 2; see p. 375.

$$\begin{aligned}
 x_1 &= x_1(u, v) \equiv a_1 + \frac{2a|k|\operatorname{Re}f(z)}{|f(z)|^2 + k^2}, \\
 x_2 &= x_2(u, v) \equiv a_2 + \frac{2a|k|\operatorname{Im}f(z)}{|f(z)|^2 + k^2}, \\
 x_3 &= x_3(u, v) \equiv k \left[1 - \frac{2a|k|}{|f(z)|^2 + k^2} \right],
 \end{aligned}
 \tag{8}$$

where $f(z)$ has one of the following forms:

$$f(z) \equiv \alpha z + \beta, \quad f(z) \equiv \alpha \bar{z} + \beta,$$

where α and β are constants. In (8), $k = a_3 \pm a$, $|k|$ is the maximum of the two quantities $|a_3 + a|$ and $|a_3 - a|$ and (a_1, a_2, a_3) are the co-ordinates of the center of S .

From 2.1–2.3, it follows that we may assume $a_1 = a_2 = a_3 = 0$, $k = a$, in (8), and that we may assume the function $f(z)$ is given by

$$f(z) \equiv \alpha z, \quad \alpha > 0. \tag{9}$$

This is equivalent to assuming that the point $z = \infty$ corresponds to the point $P: (0, 0, a)$ on S , that the point $z = 0$ corresponds to the point $P': (0, 0, -a)$ which is diametrically opposite P , and that the point in the z -plane corresponding to $(a, 0, 0)$ is real and positive. Since we are investigating *all* mean-value surfaces associated with S , it follows from 2.4 that we may take $\alpha = 1$ in (9), in which case the functions (1) have the following familiar representation, as given by (8):

$$\begin{aligned}
 x_1 &= x_1(u, v) \equiv \frac{2a^2u}{u^2 + v^2 + a^2}, \\
 x_2 &= x_2(u, v) \equiv \frac{2a^2v}{u^2 + v^2 + a^2}, \\
 x_3 &= x_3(u, v) \equiv a \left[1 - \frac{2}{u^2 + v^2 + a^2} \right].
 \end{aligned}
 \tag{10}$$

If C_r is the circle $u^2 + v^2 = r^2$, and if

$$z = u + iv = re^{i\theta},$$

then (4) and (10) yield

$$\begin{aligned}
 A_{1,\rho}(u, v) + iA_{2,\rho}(u, v) &= e^{i\theta} [A_{1,\rho}(r, 0) + iA_{2,\rho}(r, 0)], \\
 A_{3,\rho}(u, v) &= A_{3,\rho}(r, 0),
 \end{aligned}$$

from which it follows that the map of C_r on the mean-value surface T_ρ , associated with the sphere S which is defined by (10), is a circle C_r^* in a plane perpendicular to the x_3 -axis; moreover, the center of C_r^* is on the x_3 -axis. Since S_ρ lies on T_ρ , it follows that the functions (4) map D_ρ isothermally on a surface that lies on the surface of revolution T_ρ .

In §4 it will appear that for $0 < \rho < \infty$, T_ρ is not a sphere. Nevertheless, we shall call T_ρ a *mean-value sphere*.

4. Mean-value spheres. The mean-value sphere T_ρ is a surface of revolution about the x_3 -axis. Accordingly, to investigate T_ρ , it is sufficient to study the intersection T_ρ^* of T_ρ with the plane $x_2 = 0$. Since, by (10),

$$x_2(u, -v) \equiv -x_2(u, v),$$

it follows that the intersection of T_ρ with the plane $x_2 = 0$ can be obtained from (4) by setting $v = 0$ in (10). A computation yields the following coordinate functions for T_ρ^* :

$$\begin{aligned} x_1 &= A_{1,\rho}(u, 0) \\ &\equiv 4a^2u/[a^2 + \rho^2 + u^2 + ((a^2 + \rho^2 - u^2)^2 + 4a^2u^2)^{1/2}], \\ (11) \quad x_3 &= A_{3,\rho}(u, 0) \\ &\equiv a - \frac{2a^3}{\rho^2} \log \frac{a^2 + \rho^2 - u^2 + ((a^2 + \rho^2 - u^2)^2 + 4a^2u^2)^{1/2}}{2a^2}. \end{aligned}$$

We make the following observations.

4.1. The curve T^* is symmetric about the x_3 -axis; since

$$\begin{aligned} A_{1,\rho}(u, 0) &\equiv A_{1,\rho}\left(\frac{u^2 + \rho^2}{u}, 0\right), \\ A_{3,\rho}(u, 0) &\equiv 2a - \frac{2a^3}{\rho^2} \log \frac{a^2 + \rho^2}{a^2} - A_{3,\rho}\left(\frac{u^2 + \rho^2}{u}, 0\right), \end{aligned}$$

it follows that T_ρ^* is also symmetric about the line

$$(12) \quad x_3 = A_{3,\rho}((a^2 + \rho^2)^{1/2}, 0) = a - \frac{a^3}{\rho^2} \log \frac{a^2 + \rho^2}{a^2}.$$

From this symmetry and from the relations,

$$m = \frac{2au}{a^2 + \rho^2 - u^2}, \quad \frac{dm}{du} = \frac{2a(a^2 + \rho^2 + u^2)}{(a^2 + \rho^2 - u^2)^2},$$

satisfied by the slope m , it follows that T_ρ^* is convex.

4.2. The height h and the width l of T_ρ^* are given by

$$h = \frac{2a^3}{\rho^2} \log \frac{a^2 + \rho^2}{a^2}, \quad l = \frac{4a^2}{(a^2 + \rho^2)^{1/2} + a};$$

hence for $0 < \rho < \infty$ the curve T_ρ^* is not a circle. Moreover, since

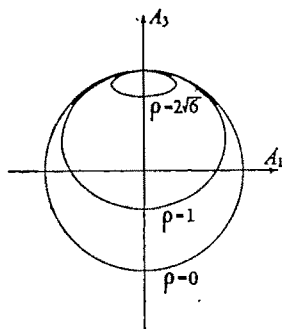
$$\lim_{\rho \rightarrow \infty} \frac{l}{h} = \infty,$$

it follows that T_ρ^* "flattens out" while approaching the point $(0, a)$, as $\rho \rightarrow \infty$.

4.3. Each member of the family $[T_\rho^*]$ passes through the point $(0, a)$ and is tangent there to every other member of the family. For $\rho = 0$, T_ρ^* is the circle $x_1^2 + x_3^2 = a^2$, and for $\rho = \infty$, T_ρ^* is the point $(0, a)$.

4.4. A computation shows that for $0 \leq \rho < \rho'$, T_ρ^* is inside $T_{\rho'}^*$, except for their common point of tangency.

The figure shows T_ρ^* for $\rho = 0, 1, 2(6)^{1/2}, \infty$; $a = 1$.



From 4.1–4.4 it follows that the family of surfaces $[T_\rho]$ consists of convex surfaces each of which passes through the point $(0, 0, a)$ in (x_1, x_2, x_3) -space and is tangent there to every other member of the family. The surface T_ρ is a surface of revolution about the x_3 -axis and is symmetric with respect to the plane

$$x_3 = a - \frac{a^3}{\rho^2} \log \frac{a^2 + \rho^2}{a^2}.$$

Since the ratio of width to height $\rightarrow \infty$ as $\rho \rightarrow \infty$, it follows that T_ρ "flattens out" as $\rho \rightarrow \infty$. For $\rho = 0$, T_ρ is the sphere about the origin with radius a , and for $\rho = \infty$, T_ρ is the point $(0, 0, a)$. For $0 \leq \rho' < \rho$, T_ρ is inside $T_{\rho'}$, except for their common point of tangency. From (11) we obtain the following isothermic representation for T_ρ :

$$x_1 = A_{1,\rho}(u, v)$$

$$\equiv \frac{a^2 u}{\rho^2(u^2 + v^2)} [a^2 + u^2 + v^2 + \rho^2 - ((a^2 + \rho^2 - u^2 - v^2)^2 + 4a^2(u^2 + v^2))^{1/2}],$$

$$x_2 = A_{2,\rho}(u, v)$$

$$\equiv \frac{a^2 v}{\rho^2(u^2 + v^2)} [a^2 + u^2 + v^2 + \rho^2 - ((a^2 + \rho^2 - u^2 - v^2)^2 + 4a^2(u^2 + v^2))^{1/2}],$$

$$x_3 = A_{3,\rho}(u, v)$$

$$\equiv a - \frac{2a^3}{\rho^3} \log \left[\frac{a^2 - u^2 - v^2 + \rho^2 + ((a^2 + \rho^2 - u^2 - v^2)^2 + 4a^2(u^2 + v^2))^{1/2}}{2a^2} \right].$$

THE OHIO STATE UNIVERSITY AND
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NOTE ON THE DISTRIBUTION OF VALUES OF THE ARITHMETIC FUNCTION $d(m)^1$

M. KAC

1. **Introduction.** Recently Dr. Erdős and the present writer² proved the following theorem:

If $\nu(m)$ denotes the number of different prime divisors of m and $k_n(\omega)$ the number of positive integers $m \leq n$ for which

$$\nu(m) \leq \lg \lg n + \omega(2 \lg \lg n)^{1/2},$$

then

$$\lim_{n \rightarrow \infty} \frac{k_n(\omega)}{n} = \pi^{-1/2} \int_{-\infty}^{\omega} e^{-u^2} du = D(\omega).$$

The purpose of this note is to derive a similar theorem concerning the function $d(m)$ which denotes the number of all different divisors of m (1 and m are included).

In fact we are going to prove the following theorem:

If $r_n(\omega)$ denotes the number of positive integers $m \leq n$ for which

$$d(m) \leq 2 \lg \lg n + \omega(2 \lg \lg n)^{1/2},$$

¹ Presented to the Society, May 2, 1941.

² P. Erdős and M. Kac, *The Gaussian law of errors in the theory of additive number theoretic functions*, American Journal of Mathematics, vol. 62, pp. 738-742.

then

$$\lim_{n \rightarrow \infty} \frac{r_n(\omega)}{n} = \pi^{-1/2} \int_{-\infty}^{\omega} e^{-u^2} du = D(\omega).$$

2. **Proof of the theorem.** The proof is based on the theorem cited in the introduction and on the following two facts:

I. The mean value

$$M\{d(m)/2^{r(m)}\} = \lim_{n \rightarrow \infty} n^{-1} \sum_{m=1}^n d(m)/2^{r(m)}$$

exists and is finite.

II. If $f(m) \geq 0$ is such that $M\{f(m)\}$ is finite, if $\lim_{n \rightarrow \infty} g(n) = \infty$ as $n \rightarrow \infty$ and if $p(n)$ denotes the number of positive integers $m \leq n$ for which $f(m) \leq g(n)$, then $\lim_{n \rightarrow \infty} p(n)/n = 1$ as $n \rightarrow \infty$.

I is implied by a theorem of E. R. van Kampen and A. Wintner³ and II is almost obvious even under a weaker condition that $\limsup_{n \rightarrow \infty} n^{-1} \sum_{m=1}^n f(m) < \infty$.

Let ω be an arbitrary real number and $\epsilon > 0$. Put $f(m) = d(m)/2^{r(m)}$ and $g(n) = 2^{(2 \lg \lg n + (\omega - \epsilon)(2 \lg \lg n)^{1/2})^{1/3}}$. Let F_n be the set of positive integers $m \leq n$ for which $v(m) \leq \lg \lg n + (\omega - \epsilon)(2 \lg \lg n)^{1/2}$, G_n the set of positive integers $m \leq n$ for which $f(m) \leq g(n)$ and H_n the set of positive integers $m \leq n$ for which $d(m) \leq 2^{1 \lg \lg n + \omega(2 \lg \lg n)^{1/2}}$. If $m \in F_n G_n$ then $m \in H_n$. Hence,

$$F_n G_n \subset H_n.$$

The number of elements in F_n is $k_n(\omega - \epsilon)$; in G_n , $p(n)$; and in H_n , $r_n(\omega)$.

Thus, the number of elements in $F_n G_n$ is $\geq k_n(\omega - \epsilon) - (n - p(n))$ and finally

$$k_n(\omega - \epsilon) - (n - p(n)) \leq r_n(\omega).$$

On the other hand for every m , $2^{r(m)} \leq d(m)$ (the equality occurs only if m is a prime) and therefore $H_n \subset F_n$ or $r_n(\omega) \leq k_n(\omega)$. The inequalities combined give

$$k_n(\omega - \epsilon) - (n - p(n)) \leq r_n(\omega) \leq k_n(\omega).$$

But as $n \rightarrow \infty$ $k_n(\omega - \epsilon)/n \rightarrow D(\omega - \epsilon)$, $k_n(\omega)/n \rightarrow D(\omega)$ and $(n - p(n))/n \rightarrow 0$ (see I and II); hence

$$D(\omega - \epsilon) \leq \liminf_{n \rightarrow \infty} \frac{r_n(\omega)}{n} \leq \limsup_{n \rightarrow \infty} \frac{r_n(\omega)}{n} \leq D(\omega).$$

³ American Journal of Mathematics, vol. 62, p. 618 (Theorem IV).

Since ϵ is arbitrary and $D(\omega)$ is a continuous function of ω ,

$$\lim_{n \rightarrow \infty} \frac{r_n(\omega)}{n} = D(\omega).$$

3. **Some numerical results.** The values of $d(m)$ for $1 \leq m \leq 10^4$ can be found in the recent tables of the British Association for the Advancement of Science,⁴ so that it was easy to obtain the exact value of $10^{-4}r_{10^4}(\omega)$ for different values of ω and compare them with the values of $D(\omega)$ computed from the tables of the probability integral.⁵

$2 \lg \lg n + \omega (2 \lg \lg n)^{1/2}$	ω	$10^{-4}r_{10^4}(\omega)$	$D(\omega)$
1	-1.054	0.0001	0.0680
2	-0.579	0.1230	0.3065
3	-0.302	0.1255	0.4346
4	-0.105	0.3863	0.4410
5	0.048	0.3867	0.5220
6	0.173	0.4631	0.5961
7	0.279	0.4633	0.6534
8	0.370	0.6747	0.6996
9	0.451	0.6779	0.7382
10	0.523	0.6929	0.7702
11	0.588	0.7970	0.7971
12	0.648	0.7971	0.8202
13	0.702	0.8012	0.8396
14	0.753	0.8027	0.8565
15	0.800	0.8827	0.8710
16	0.845	0.8827	0.8840

For a good agreement $n = 10,000$ seems to be too small. The rather striking fact that $10^{-4}r_{10^4}(.588)$ is almost equal to $D(.588)$ is probably accidental. The case $\omega = .800$ disproves the conjecture that always $n^{-1}r_n(\omega) \leq D(\omega)$.

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⁴ *Mathematical Tables*, vol. 8, *Number-Divisor Tables*, Cambridge University Press, 1940. We refer in particular to Table III. In these tables $d(m)$ is denoted by $\nu(m)$.

⁵ We used the tables on pages 388 to 391 of the first volume of Czuber's *Wahrscheinlichkeitsrechnung*, Teubner, 1908. It should be noted that $D(\omega) = \{1 + \Phi(\omega)\}/2$.

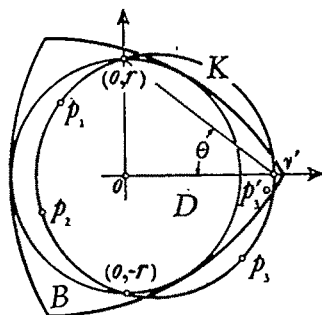
We also wish to thank Mr. W. J. Harrington for his help in computing the table given on this page.

A CHARACTERIZATION OF THE DISC

C. V. ROBINSON¹

In this paper the disc² is characterized as the *only* connected, simply connected domain³ B with the following property, C_3 : B will cover (by an isometry) any subset P of the plane provided B will cover each 3 points of P .

The disc has property C_3 . For a plane set P can be covered by a ρ -disc if and only if the members of the family F of ρ -discs with centers in P have a common point. If now each three points of P are



on a ρ -disc then each three discs of F intersect and by a theorem on convex bodies due to E. Helly⁴ there is a point common to all the discs of F .

LEMMA. *A bounded, closed subset of the plane contains a largest circle.*

The proof is accomplished by selecting a sequence of circles from the set whose centers converge to a point and whose radii converge to the least upper bound of the radii of circles in the set and (using the fact that the set is closed) showing that the limiting circle belongs to the set.

THEOREM. *The disc is the only connected, simply connected domain with property C_3 .*

PROOF. We assume B to be the given domain, and show that B

¹ Part of a Ph.D. dissertation at University of Missouri, under L. M. Blumenthal, 1940.

² The disc of radius ρ and center p is the set of points x of E_2 such that $px \leq \rho$.

³ Closure of a bounded open subset of E_2 .

⁴ Theorem. If each $n+1$ sets of a family of *bounded, closed, convex* subsets of E_n intersect, there is a point common to all the sets. Jahrbuch der Deutschen Mathematiker-Vereinigung, vol. 32 (1932), pp. 175-176.

must be a disc. By the preceding lemma, B contains a largest circle (center o and radius r), and since B is simply connected, the whole disc D of the circle is contained in B . If we suppose B is not a disc, there is a point p of B outside D , and, since B is connected, a point v' of B such that $r < ov' \leq \max(op, 3^{1/2}r)$. (The inequality on the right insures that $\theta \geq 30^\circ$ in the figure.) Introduce coordinates in the plane with o as origin and the ray ov' as positive X -axis. Then D is defined by the inequality

$$(1) \quad D: x^2 + y^2 \leq r^2,$$

and the circle K of the three points $(0, \pm r)$, v' has the equation

$$(2) \quad K: \left(x - \left(\frac{v^2 - r^2}{2v} \right) \right)^2 + y^2 = \left(\frac{v^2 + r^2}{2v} \right)^2$$

where v is the abscissa of the point v' .

The circle K has the following properties:

(α) The radius of K is greater than r . For, since $v > r$,

$$\left(\frac{v^2 + r^2}{2v} \right) = \left(\frac{v^2 - r^2}{2v} \right)^2 + r^2 > r^2.$$

(β) The arc of K to the left of the Y -axis is of length $\geq 120^\circ$. For the arc is $2(2\theta) = 4\theta \geq 120^\circ$ by choice of v' .

(γ) The arc of K to the left of the Y -axis is in D . From (2)

$$(3) \quad x^2 + y^2 = r^2 + x \left(\frac{v^2 - r^2}{v} \right)$$

so that for $x \leq 0$ the inequality (1) is satisfied by points of K .

(δ) The maximum distance of a point of K from the origin is v . This follows easily from (3).

If, now, any three points p_1, p_2, p_3 of K are selected, at least two of them, say p_1, p_2 , will lie on an arc of length $\leq 120^\circ$. By property (β), K can then be rotated about its center so that this arc will fall to the left of the Y -axis, that is, $p_1, p_2 \in D$ by property (γ). Since $op_3 \leq v$ by property (δ), there is, since B is connected, a point p'_3 of B at distance op_3 from o . Rotate K about the origin so that p_3 goes into p'_3 . Since p_1, p_2 will remain in D , the three points p_1, p_2, p_3 are seen to be congruent to three points of B . But p_1, p_2, p'_3 are any three points of K , so that B must, by our assumption, cover K . By property (α) this is impossible, and hence no point p of B can lie outside D , that is, B is the disc D .

ON SOME PROPERTIES OF SYMMETRICAL PERFECT SETS¹

R. SALEM

This paper deals with some properties of symmetrical perfect sets, in view of applications: (i) to the problem of multiplicity of trigonometrical series; (ii) to the study of Fourier-Stieltjes coefficients of singular monotone functions; (iii) to the problem of absolute convergence of trigonometrical series.

1. Sets of multiplicity of trigonometrical series. We shall consider, throughout the paper, symmetrical perfect sets, that is, sets which are obtained, in the closed interval $(0, 2\pi)$ by the following process. We divide the interval in three parts of lengths proportional to ξ_1 , $1 - 2\xi_1$, ξ_1 , and we remove the central open interval. In the second operation we divide each one of the two intervals left in three parts proportional to ξ_2 , $1 - 2\xi_2$, ξ_2 , and we remove the two central open intervals, and so on infinitely, the sequence $\{\xi_p\}$ being such that $0 < \xi_p \leq \frac{1}{2}$.² After p operations, we have thus removed $2^p - 1$ intervals which we shall denote by δ_{pk} ($k = 1, 2, \dots, 2^p - 1$) and 2^p intervals are left, which we shall denote by η_{pk} ($k = 1, 2, \dots, 2^p$). Each interval η_{pk} is of length equal to $\eta_p = 2\pi \cdot \xi_1 \xi_2 \cdots \xi_p$. Let E_p be the set constituted by the 2^p intervals η_{pk} and let $E(p)$ be its measure. We have

$$(1) \quad E(p) = 2^p \eta_p = 2\pi \cdot 2^p \xi_1 \xi_2 \cdots \xi_p.$$

$E(p)$ is a non-increasing function of p and the measure of the perfect set P obtained by the above-described process is $\lim E(p)$ for $p = \infty$. We shall only consider, throughout the paper, sets for which this limit is equal to zero.

We shall now construct a monotone continuous function $F(x)$ constant in every interval contiguous to P but increasing from one interval to another, by the following well known process.³ For every p let F_p be a non-decreasing continuous function defined by the following conditions: $F_p(0) = 0$, $F_p(2\pi) = 1$,

$$F_p(x) = \frac{k}{2^p} \text{ in } \delta_{pk}, \quad k = 1, 2, \dots, 2^p - 1,$$

¹ Presented to the Society, May 3, 1941.

² If $\xi_p = \frac{1}{2}$ no interval is removed in the p th operation but the intervals left after the $(p-1)$ th operation are subdivided in two equal parts, and we deal in the $(p+1)$ th operation, with the intervals thus subdivided.

³ See Menchoff [1]; Zygmund [1, p. 294].

and $F_p(x)$ linear in each η_{pk} . The consideration, in an interval η_{pk} , of the straight line representing F_p and of the broken line representing F_{p+1} shows immediately that

$$|F_p - F_{p+1}| \leq \frac{1}{2^{p+1}} \frac{\frac{1}{2}\eta_p - \eta_{p+1}}{\frac{1}{2}\eta_p};$$

that is to say, by (1)

$$(2) \quad |F_p - F_{p+1}| \leq \frac{1}{2^{p+1}} \left[1 - \frac{E(p+1)}{E(p)} \right].$$

Thus F_p converges uniformly to a continuous function F which is non-decreasing, is constant in every interval contiguous to P but increasing from one interval to another, and we have $F(0) = 0$, $F(2\pi) = 1$.

Let us consider now the Fourier-Stieltjes coefficients of dF . We have

$$c_n = \int_0^{2\pi} e^{nix} dF = \int_0^{2\pi} e^{nix} d(F - F_p) + \int_0^{2\pi} e^{nix} dF_p.$$

Denoting by A and B the two last integrals, and integrating A by parts, we have, as $F - F_p$ vanishes for $x=0$ and $x=2\pi$,

$$|A| \leq n \int_0^{2\pi} |F - F_p| dx.$$

Hence

$$|A| \leq n \int_0^{2\pi} |F_p - F_{p+1}| dx + n \int_0^{2\pi} |F_{p+1} - F_{p+2}| dx + \dots$$

But

$$\int_0^{2\pi} |F_p - F_{p+1}| dx \leq E(p) \cdot \max |F_p - F_{p+1}|,$$

since $F_p = F_{p+1}$ when x does not belong to E_p . Hence, by (2)

$$\int_0^{2\pi} |F_p - F_{p+1}| dx \leq \frac{1}{2^{p+1}} [E(p) - E(p+1)].$$

Let us suppose now that $E(p)$ is a convex function; then $E(p) - E(p+1)$ is decreasing and we have

$$|A| \leq n[E(p) - E(p+1)] \sum_{p=p}^{\infty} \frac{1}{2^{p+1}} = \frac{n}{2^p} [E(p) - E(p+1)].$$

Considering now the integral B we have

$$\int_{\mathfrak{P}_k} e^{nix} dF_p = \frac{1}{2^p \eta_p} \int_{\mathfrak{P}_k} e^{nix} dx = \frac{1}{E(p)} \int_{\mathfrak{P}_k} e^{nix} dx.$$

Hence

$$|B| \leq \frac{2^p}{E(p)} \cdot \frac{2}{n}$$

and

$$(3) \quad |c_n| \leq \frac{n}{2^p} [E(p) - E(p+1)] + \frac{2^{p+1}}{n} \frac{1}{E(p)}.$$

Up to here we have followed very closely the above quoted Menchoff's example, in extending it to general symmetrical sets. In Menchoff's example

$$\xi_p = \frac{p}{2(p+1)}, \quad E(p) = \frac{1}{p+1}.$$

Now the inequality (3) gives us in a very simple way a class of symmetrical perfect sets of measure zero which are sets of multiplicity. As is well known, in order that P should be a set of multiplicity it is sufficient to have $c_n = o(1)$.

The integer p being arbitrary, let us choose it satisfying the following condition:

$$(4) \quad \frac{2^{p-1}}{(E(p-1))^{1/2}(E(p-1) - E(p))^{1/2}} < n \leq \frac{2^p}{(E(p))^{1/2}(E(p) - E(p+1))^{1/2}}.$$

Thus, by (3)

$$|c_n| \leq \left(1 - \frac{E(p+1)}{E(p)}\right)^{1/2} + 4 \frac{E(p-1)}{E(p)} \left(1 - \frac{E(p)}{E(p-1)}\right)^{1/2}.$$

If n increases infinitely, so does p defined by (4); hence if we suppose $E(p+1)/E(p) \rightarrow 1$, we have $c_n = o(1)$.

We have thus obtained the following result:⁴ *Each symmetrical perfect set of measure zero such that $E(p)$ is convex and $E(p+1)/E(p)$ tends to 1 is a set of multiplicity.*

⁴ A more general result is due to Nina Bary [1], but the proof is very complicated.

2. The Fourier-Stieltjes coefficients of singular monotone functions. Let us now consider for every p the characteristic function f_p of the set E_p . We have

$$\int_0^{2\pi} e^{nix} dF_p = \frac{1}{2^p \eta_p} \int_{E_p} e^{nix} dx = \frac{1}{E(p)} \int_0^{2\pi} f_p(x) e^{nix} dx.$$

Hence

$$c_n = \int_0^{2\pi} e^{nix} dF = \frac{1}{E(p)} \int_0^{2\pi} f_p(x) e^{nix} dx + A,$$

A denoting, as above, the integral

$$\int_0^{2\pi} e^{nix} d(F - F_p).$$

We have seen that

$$|A| \leq n \sum_{\nu=p}^{\infty} \frac{1}{2^{\nu+1}} [E(\nu) - E(\nu+1)].$$

Here we shall make no hypothesis whatever about the behaviour of $E(p) - E(p+1)$. It is plain that in any case we have

$$|A| \leq \frac{nE(p)}{2^{p+1}}.$$

Hence

$$(5) \quad E^2(p) |c_n|^2 \leq 2 \left| \int_0^{2\pi} f_p(x) e^{nix} dx \right|^2 + 2 \left(\frac{nE^2(p)}{2^{p+1}} \right)^2.$$

The integer p is arbitrary. We can associate to every n an integer $p = p(n)$ non-decreasing, tending to infinity with n , and sum the inequalities thus obtained from $n=1$ to $n=\infty$ provided that the sum of the right-hand side terms be convergent.

Let us observe that the sets E_p are such that $E_{p+1} \subset E_p$ that is to say $f_{p+1} \leq f_p$. Now it is known that in this case the sum

$$S = \sum_1^{\infty} \left| \int_0^{2\pi} f_p(x) e^{nix} dx \right|^2,$$

where p takes the same value for all n such that $n_p \leq n < n_{p+1}$, converges if $n_{p+1}/n_p > \lambda > 1$; moreover $S < C(\lambda)$, C being a constant depending only on λ . This is a consequence of the well known theorem stating that if $S_n(x)$ is the n th sum of the Fourier series of a function ϕ of integrable square, then

$$\int_0^{2\pi} \sup_p |S_{n_p}(x)| dx \leq K(\lambda) \left[\int_0^{2\pi} \phi^2 dx \right]^{1/2}$$

where $K(\lambda)$ depends on λ only.⁵

Hence if we take, for example $p = [a \log n]$ ($[m]$ denoting the integral part of m), p will take the same value for $n_p \leq n < n_{p+1}$, n_p being given by

$$a \log (n_p - 1) < p \leq a \log n_p.$$

Hence

$$a \log n_{p+1} - a \log (n_p - 1) > 1;$$

that is to say,

$$\frac{n_{p+1}}{n_p} > \lambda > 1$$

λ depending on a only.

On the other hand, we will have

$$\frac{n}{2^{p+1}} < \frac{n}{2^{a \log n}}$$

and it is easily seen that the series $\sum (n/2^{p+1})^2$ will converge if $a > 3/(2 \log 2)$.

Hence, returning to inequality (5) and writing $E[a \log n]$ instead of $E([a \log n])$, we get the following result:

The series

$$T = \sum_1^\infty E^2[a \log n] \cdot |c_n|^2$$

is convergent for $a > 3/(2 \log 2)$. Moreover $T < C(a)$, C being a constant depending on a only.

⁵ See Zygmund [1, p. 252]; and for the equivalence of the two statements, Salem [1, 2]. The proof is briefly as follows: $p = p(x)$ being any function of x , we have

$$\left| \int_0^{2\pi} S_{n_{p(x)}}(x) dx \right| \leq K(\lambda) \left[\int_0^{2\pi} \phi^2 dx \right]^{1/2},$$

and this can be written if $\phi(x) \sim \sum c_n e^{nix}$,

$$\left| \sum c_n \int_0^{2\pi} f_p e^{nix} dx \right| \leq K_1(\lambda) (\sum |c_n|^2)^{1/2},$$

where the f_p are characteristic functions of sets, corresponding to $p(x)$, such that $f_{p+1} \leq f_p$ and such that f_p is the same for all n for which $|n_p| \leq |n| < |n_{p+1}|$. Since this holds good for any sequence $\{c_n\}$ such that $\sum |c_n|^2 < \infty$, the statement about the sum S follows immediately.

The interest of this theorem lies in the fact that we can construct our perfect set P of measure zero such that $E(p)$ tends to zero as slowly as we please. Hence, while the function $F(x)$'s being singular (that is, constant in every interval contiguous to P) makes the series $\sum |c_n|^2$ necessarily divergent, the divergence of this series can be as slow as we please, and the rapidity of the divergence is connected in a very simple way with the function $E(p)$.

We refer the reader to other results which have been obtained previously in this field. See Wiener and Wintner [1, 2] and Schaeffer [1].

3. Absolute convergence of trigonometrical series. The set P is said to be of the type N if there exists a trigonometrical series $\sum \rho_n \cos (nx - \alpha_n)$ with $\rho_n \geq 0$, $\sum \rho_n = \infty$, such that $\sum |\rho_n| |\cos (nx - \alpha_n)| < \infty$ for every x belonging to P . It has been proved⁶ that if P is of the type N we have

$$\limsup \int_0^{2\pi} \cos 2nx \, dF = 1.$$

If we split the sequence of integers $\{n\}$ in two complementary sequences $\{m_k\}$, $\{n_k\}$ such that for every k :

$$(6) \quad \int_0^{2\pi} \cos 2m_k x \, dF \leq 1 - \epsilon, \quad \int_0^{2\pi} \cos 2n_k x \, dF > 1 - \epsilon,$$

ϵ being any fixed positive number < 1 , the proof of the theorem just stated shows that no series $\sum r_k \cos (m_k x - \alpha_k)$ can converge absolutely in P unless $\sum r_k < \infty$. Hence, if the set P is of the type N every trigonometrical series absolutely convergent in P but not everywhere can be broken into two complementary series: the first one,

$$\sum r_k \cos (m_k x - \alpha_k) \quad \text{with } \sum r_k < \infty,$$

which is a parasitic series convergent absolutely everywhere; and the second one, $\sum \rho_k \cos (n_k x - \beta_k)$, which converges absolutely in P , with $\sum \rho_k = \infty$.

Now, the question arises of what can be the frequency of the integers $\{n_k\}$. We can draw some conclusions from the theorem proved in §2. We have, by (6), by a suitable choice of ϵ , $|c_{2n_k}| > \frac{1}{2}$. Hence

$$\sum_1^\infty E^2[a \log 2n_k] < \infty.$$

⁶ See Salem [3, p. 323].

Roughly speaking, the more slowly $E(p)$ decreases, the more sparse are the integers $\{n_k\}$. More precisely if between p and q there are $\nu(p, q)$ integers of the sequence $\{n_k\}$ we have

$$\nu \cdot E^3[a \log 2q] < \sum_{p < n_k < q} E^3[a \log 2n_k] < E[a \log 2p] \cdot C(a);$$

hence

$$(7) \quad \nu < C(a) \cdot \frac{E[a \log 2p]}{E^3[a \log 2q]}.$$

It has been shown⁷ that the perfect set P for which all the ξ_p are equal to $\frac{1}{2}$, except for a sequence $\{i_j\}$ for which $\xi_{i_j} = 1/2j$, is of the type N . By taking the sequence $\{i_j\}$ very sparse, we can plainly obtain a set of the type N with $E(p)$ decreasing as slowly as we please. Hence, the inequality (7) shows that there exist sets of the type N for which $\nu[p, \phi(p)] < \omega(p)$, $\phi(p)$ increasing as rapidly, and $\omega(p)$ as slowly as we please.

This result throws some light on the fact that the sum of two sets P, P' , both of the type N , may be not of the type N , a result which is due to Marcinkiewicz.⁸ If the sets P, P' are both of the type N , then, F' being the corresponding function for the set P' , there exist two sequences of integers $\{n_k\}, \{n'_k\}$ such that

$$(8) \quad \lim \int_0^{2\pi} \sin^2 n_k x dF = 0, \quad \lim \int_0^{2\pi} \sin^2 n'_k x dF' = 0$$

by the theorem stated at the beginning of §3. But it can be shown exactly in the same way that if $P+P'$ is of the type N , we must have

$$\liminf \left[\int_0^{2\pi} \sin^2 nx dF + \int_0^{2\pi} \sin^2 nx dF' \right] = 0;$$

that is to say, there must exist a *common* sequence $\{m_k\}$ such that

$$(9) \quad \lim \int_0^{2\pi} \sin^2 m_k x dF = 0, \quad \lim \int_0^{2\pi} \sin^2 m_k x dF' = 0.$$

Now the sequences $\{n_k\}, \{n'_k\}$ satisfying (8) can be, as we have seen, both very sparse, hence the existence of a common sequence $\{n_k\}$ satisfying (9) may be impossible.

The question of what happens if the second set P' is reduced to a

⁷ See Salem [3, p. 329].

⁸ Marcinkiewicz [1].

single point has been mentioned to me by Dr. Marcinkiewicz and Professor Zygmund. We will show here that *the sum of any set of the type N (not necessarily a perfect set) and of a single point is also of the type N .*

It has been proved³ that by a suitable translation of the given set E we can suppose that the series converging absolutely in E is a sine series. Let us suppose, then, that for every x belonging to E we have

$$\sum_1^{\infty} \rho_n |\sin nx| < \infty, \quad \rho_n \geq 0, \quad \sum \rho_n = \infty.$$

We can always find a positive function $\omega(n)$ increasing infinitely with n , such that

$$\sum_1^{\infty} \frac{\rho_n}{\omega(n)} = \infty, \quad \sum_1^{\infty} \frac{\rho_n}{\omega^2(n)} < \infty.$$

(We can take, for example, $\omega(n) = \sum_1^n \rho_n$.) Let Q be a point of abscissa x_0 , not belonging to E . By Dirichlet's theorem, for every n we can find an integer p_n such that

$$(10) \quad 1 \leq p_n \leq \omega(n), \quad \left| p_n \left(n \frac{x_0}{\pi} \right) - h_n \right| < \frac{1}{[\omega(n)]},$$

p_n being an integer and $(z) = z - [z]$ denoting the fractional part of z . Thus

$$\left| p_n n \frac{x_0}{\pi} - p_n \left[n \frac{x_0}{\pi} \right] - h_n \right| < \frac{1}{[\omega(n)]};$$

that is to say,

$$|p_n n x_0 - k_n \pi| < \frac{4}{\omega(n)},$$

k_n being an integer. Thus

$$|\sin p_n n x_0| < \frac{4}{\omega(n)}$$

for every n . Hence

$$(11) \quad \sum_1^{\infty} \frac{\rho_n}{\omega(n)} |\sin p_n n x_0| < 4 \sum_1^{\infty} \frac{\rho_n}{\omega^2(n)} < \infty.$$

³ See Salem [3, p. 319].

On the other hand, applying the well known theorem: $|\sin mt| \leq m |\sin t|$ for m a positive integer, we have, by (10), if x belongs to E :

$$(12) \quad \sum_{n=1}^{\infty} \frac{\rho_n}{\omega(n)} |\sin p_n n x| \leq \sum_{n=1}^{\infty} \frac{p_n}{\omega(n)} \rho_n |\sin n x| \\ \leq \sum_{n=1}^{\infty} \rho_n |\sin n x| < \infty.$$

Hence, by (11) and (12) the series $\sum_{n=1}^{\infty} [\rho_n/\omega(n)] \sin nx$ converges absolutely in $E+Q$. But $\sum_{n=1}^{\infty} \rho_n/\omega(n) = \infty$; hence $E+Q$ is of the type N .

This leads immediately to the following result: *The sum of any set of the type N and of any finite set is also of the type N .*

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MONTREAL, QUEBEC, CANADA





**REPORT OF THE WAR PREPAREDNESS COMMITTEE OF
THE AMERICAN MATHEMATICAL SOCIETY AND
MATHEMATICAL ASSOCIATION OF AMERICA
AT THE CHICAGO MEETING**

This report is concerned with three important questions: (1) The deferment of mathematicians; (2) Report of Subcommittee on Education for Service; (3) Use of Selectees.

1. THE DEFERMENT OF MATHEMATICIANS

When this problem arose for the first time over a year ago the great majority of mathematicians were against seeking any special consideration of mathematicians in connection with the Selective Service. It is the feeling of many that the situation has now changed to such an extent that steps should now be taken leading to the "individual occupational deferment" of mathematicians.

We are concerned in the first place with teachers of mathematics in the colleges and universities, with advanced students of mathematics, and with mathematicians in industry. Mathematicians in the industries essential for the defense are fairly well taken care of in accordance with recommendations to the Selective Service Boards by their employers. But teachers of mathematics are variously treated by the Selective Service Boards. Some men are deferred because they are teaching mathematics to engineers, but this rule is not general. A state Selective Service official to whom the case of a mathematician was appealed, stated that the national authorities had certified no shortage of mathematicians.

Because of the Selective Service law each man must be treated on his own merits in accordance with the principles of "individual occupational deferment." It became apparent on consultation with Dr. Zook of the American Council on Education and others that the opinions of mathematicians must be given statistical support. A subcommittee of the War Preparedness Committee on the Supply and Demand for Mathematicians was accordingly appointed in June as follows: Professors T. Y. Thomas (chairman), Harry Bateman, E. T. Bell, G. C. Evans, and W. M. Whyburn. This committee has begun its investigations. It has sent a questionnaire to the heads of departments of mathematics. The committee is concerned with the number, relative quality, induction, and employment of mathematicians of the Selective Service age. It will give special attention to the needs of engineering schools and industry.

If the results shown by this questionnaire warrant, a report should be made to General Hershey in charge of the Selective Service. Late last fall the American Council on Education issued a bulletin backed by General Hershey in which local Selective Service Boards were advised to give special consideration to men in chemistry, physics, engineering, biology, et cetera, with reference to the desirability of granting "individual occupational deferment." Mathematics was not listed. This bulletin was prepared with the aid of President Jewett of the National Academy.

This spring President Morse and Dean Richardson recommended to President Jewett that applied mathematics and, in particular, engineering mathematics be added to the fields listed in the bulletin of the American Council on Education.

In conclusion we may say that the opinion of mathematicians that there is a shortage or potential shortage of mathematicians will not affect the Selective Service policy unless backed by an established statistical position and a united front.

2. SUBCOMMITTEE ON EDUCATION FOR SERVICE

An excellent report by this committee has been published in the *American Mathematical Monthly*. The earlier joint report with Morse was published in the *Monthly* and in the *Mathematics Teacher*. Copies were bought by the Government in some numbers. Considerable interest has been shown by various schools and teachers associations. The Hart report has reached a large number of college and university teachers of mathematics and some high school teachers. Nevertheless it is felt desirable that it also be brought to the attention of administrators of secondary schools.¹

3. ON THE USE OF SELECTEES

A considerable number of young mathematicians write to Morse as Chairman of the War Preparedness Committee stating that they may be selected and asking for help. Some wish to be used outside the Selective Service, others merely to be used in the Selective Service in some way appropriate to their mathematical training.

Very little can be done to obtain employment for these men outside the Selective Service. However, in all cases a letter has been sent to the Adjutant General giving essential data concerning the selectee together with recommendations as to his use when he is especially well qualified. These letters are cordially received by the Adjutant Gen-

¹ The Council and Board of Governors subsequently voted to distribute the report to the list of 3500 superintendents of schools of the National Education Association.

eral who has prescribed the general nature of the data desired. These data include a brief tabular vita, with age, degrees, positions, experience of mathematician and number and address of local Selective Service Board. Courses in engineering and experience with precision instruments are important. A number of the better trained young mathematicians have been placed in positions where their talents will be used to advantage.

MARSTON MORSE,
Chairman, War Preparedness Committee

THE SUMMER MEETING IN CHICAGO

The forty-seventh Summer Meeting of the Society and the twenty-third Colloquium were held at the University of Chicago, Chicago, Illinois, Tuesday to Saturday, September 2-6, 1941. The Mathematical Association of America met on Monday morning and afternoon, and jointly with the Society on Wednesday afternoon. The Institute of Mathematical Statistics held sessions Tuesday morning and afternoon and Wednesday afternoon. The Econometric Society met Tuesday, Wednesday and Thursday mornings and Tuesday afternoon.

The meeting was held in connection with the Fiftieth Anniversary Celebration of the University of Chicago, and on Tuesday morning Mr. Frederic Woodward, vice president emeritus of the University and director of the Celebration, gave a greeting and welcome to the mathematicians. As a part of the Fiftieth Anniversary Celebration, the University of Chicago collaborated with the Society in arranging a special program of lectures and conferences.

The number of papers presented at this meeting is the largest in the history of the Society and the attendance the second largest. Six hundred thirty-five persons registered including the following four hundred twenty-four members of the Society:

C. R. Adams, V. W. Adkisson, R. P. Agnew, A. A. Albert, G. E. Albert, C. B. Allendoerfer, Warren Ambrose, M. R. Anderson, T. W. Anderson, K. J. Arnold, Emil Artin, W. L. Ayres, H. M. Bacon, R. M. Ballard, J. D. Bankier, R. W. Barnard, I. A. Barnett, Walter Bartky, R. E. Basye, Harry Bateman, R. A. Beaumont, M. M. Beenken, A. A. Bennett, D. L. Bernstein, R. M. Besancon, E. E. Betz, H. R. Beveridge, S. F. Bibb, Garrett Birkhoff, G. D. Birkhoff, H. L. Black, D. H. Blackwell, H. F. Blichfeldt, G. A. Bliss, Henry Blumberg, L. M. Blumenthal, M. L. Boas, R. P. Boas, Salomon Bochner, D. G. Bourgin, J. W. Bower, M. G. Boyce, A. T. Brauer, Richard Brauer, H. E. Bray, R. W. Brink, Leonard Bristow, F. L. Brooks, H. E. Buchanan, C. E. Bures, F. J. H. Burkett, Herbert Busemann, Jewell H. Bushey, J. H. Bushey, W. H. Bussey, W. D. Cairns, J. W. Calkin, R. H. Cameron, C. C. Camp, E. J. Camp, H. H. Campaigne, A. D. Campbell, J. W. Campbell, W. B. Carver, C. R. Cassity, W. E. Cederberg, E. W. Chittenden, C. E. Clark, E. H. Clarke, M. D. Clement, A. B. Coble, C. J. Coe, L. M. Coffin, J. B. Coleman, Max Coral, J. J. Corliss, A. T. Craig, C. C. Craig, G. F. Cramer, H. B. Curry, H. B. Curtis, D. R. Curtiss, J. H. Curtiss, J. F. Daly, D. R. Davis, W. M. Davis, M. M. Day, J. J. DeCicco, L. J. Deck, F. F. Decker, C. H. Denbow, W. W. Denton, L. E. Dickson, Bernard Dimsdale, L. L. Dines, J. M. Dobbie, J. L. Doob, H. P. Doole, Arnold Dresden, D. M. Dribin, R. J. Duffin, Nelson Dunford, W. D. Duthie, P. S. Dwyer, J. J. Eachus, J. M. Earl, E. D. Eaves, M. C. Eide, Samuel Eilenberg, Churchill Eisenhart, M. L. Elveback, Benjamin Epstein, Paul Erdős, W. S. Erickson, H. J. Ettlinger, H. P. Evans, C. J. Everett, H. S. Everett, G. M. Ewing, F. A. Ficken, B. F. Finkel, C. H. Fischer, Edward Fleisher, L. R. Ford, R. M. Foster, R. H. Fox, Bernard Friedman, R. E. Fullerton, M. G. Gaba, E. R. Garbe, B. H. Gere, B. C. Getchell, K. S. Ghent, J. H. Giese, J. W. Givens, A. M. Gleason, H. H. Gold-

stine, Michael Golomb, Cornelius Gouwens, C. H. Graves, L. M. Graves, W. L. Graves, F. L. Griffin, V. G. Grove, P. E. Guenther, P. R. Halmos, R. W. Hamming, E. H. Hanson, O. G. Harrold, W. L. Hart, M. L. Hartung, J. O. Hassler, Camilla Hayden, N. A. Haynes, K. E. Hazard, E. R. Hedrick, M. H. Heins, E. D. Hellinger, M. J. Herzberger, Fritz Herzog, A. D. Hestenes, M. R. Hestenes, J. F. Heyda, J. S. Hickman, T. H. Hildebrandt, F. C. Hinds, J. J. L. Hinrichsen, G. P. Hochschild, F. E. Hohn, D. L. Holl, M. P. Hollcroft, T. R. Hollcroft, A. S. Householder, E. M. Hove, E. M. Hull, M. G. Humphreys, E. V. Huntington, D. H. Hyers, M. H. Ingraham, Dunham Jackson, R. L. Jackson, Nathan Jacobson, R. D. James, R. L. Jeffery, E. D. Jenkins, L. L. Johnson, M. M. Johnson, F. B. Jones, H. A. Jordan, Mark Kac, Shizuo Kakutani, G. K. Kalisch, Wilfred Kaplan, Irving Kaplansky, William Karush, D. E. Kearney, I. F. Keeler, J. L. Kelley, Claribel Kendall, J. F. Kenney, J. R. F. Kent, R. B. Kershner, P. W. Ketchum, J. M. Kinney, Fred Kiokemeister, C. J. Kirchen, J. R. Kline, L. A. Knowler, E. R. Kolchin, Horace Komm, C. F. Kossack, W. C. Krathwohl, M. E. Ladue, W. D. Lambert, O. E. Lancaster, E. P. Lane, R. E. Lane, R. E. Langer, C. G. Latimer, D. H. Leavens, Solomon Lefschetz, R. A. Leibler, Walter Leighton, B. A. Lengyel, C. A. Lester, P. E. Lewis, C. B. Lindquist, Marie Litzinger, Charles Loewner, M. I. Logsdon, L. A. Lorch, R. G. Lubben, C. I. Lubin, N. H. McCoy, E. A. McDougale, A. W. McGaughey, J. V. McKelvey, M. M. McKelvey, Brockway McMillan, E. J. McShane, Saunders MacLane, H. M. MacNeille, H. F. MacNeish, W. G. Madow, J. D. Mancill, Szolem Mandelbrojt, Morris Marden, C. W. Mathews, J. R. Mayor, A. E. Meder, Karl Menger, Herman Meyer, H. L. Meyer, D. D. Miller, E. W. Miller, W. L. Miser, M. G. Moore, R. L. Moore, T. W. Moore, W. L. Moore, C. W. Moran, Vladimir Morkovin, Richard Morris, Marston Morse, E. J. Moulton, D. C. Murdoch, F. H. Murray, S. B. Myers, J. P. Nash, A. L. Nelson, Albert Neuhaus, John von Neumann, C. V. Newsom, K. L. Nielsen, Ivan Niven, E. A. Nordhaus, E. B. Ogden, Rufus Oldenburger, E. G. Olds, J. M. H. Olmsted, E. J. Olson, Isaac Opatowski, Oystein Ore, F. W. Owens, H. B. Owens, J. C. Oxtoby, Gordon Pall, T. P. Palmer, R. S. Park, R. S. Pate, J. F. Paydon, C. K. Payne, A. M. Pegram, I. E. Perlin, Sam Perlis, J. W. Peters, O. J. Peterson, H. R. Phalen, R. S. Phillips, C. G. Phipps, T. A. Pierce, Everett Pitcher, George Polya, Hillel Poritsky, J. E. Powell, G. B. Price, W. T. Puckett, H. A. Rademacher, Tibor Radó, W. C. Randels, R. B. Rasmussen, Maxwell Reade, Francis Regan, P. V. Reichelderfer, W. T. Reid, Haim Reingold, Eric Reissner, C. E. Rickart, J. F. Ritt, C. V. Robinson, S. L. Robinson, V. N. Robinson, W. H. Roever, C. F. Roos, J. B. Rosenbach, R. A. Rosenbaum, Arthur Rosenthal, A. E. Ross, R. G. Sanger, Max Sasuly, A. C. Schaeffer, Henry Scheffé, M. A. Scheier, Peter Scherk, O. F. G. Schilling, Abraham Schwartz, H. M. Schwartz, G. E. Schweigert, I. E. Segal, M. E. Shanks, J. W. Sheedy, Seymour Sherman, C. G. Shover, D. R. Shreve, Harry Siller, H. A. Simmons, M. E. Sinclair, M. M. Slotnick, M. F. Smiley, A. H. Smith, E. R. Smith, F. C. Smith, H. L. Smith, R. G. Smith, Virgil Snyder, W. S. Snyder, R. H. Sorgenfrey, T. H. Southard, V. E. Spencer, A. A. Stafford, G. W. Starcher, M. E. Stark, N. E. Steenrod, B. M. Stewart, M. H. Stone, E. C. Stopher, E. B. Stouffer, W. J. Strange, Otto Szász, Gabor Szegő, A. H. Taub, M. E. Taylor, W. C. Taylor, J. M. Thompson, R. M. Thrall, W. J. Thron, L. W. Tordella, W. J. Trjitzinsky, P. L. Trump, A. W. Tucker, J. W. Tukey, A. E. Turner, S. M. Ulam, E. H. Umberger, Henry Van Engen, J. I. Vass, W. A. Vezeau, R. W. Wagner, Abraham Wald, R. J. Walker, Henry Wallman, R. M. Walter, T. O. Walton, J. H. Weaver, M. S. Webster, J. V. Wehausen, M. J. Weiss, B. A. Welch, C. P. Wells, E. T. Welmers, I. L. Wentz, M. E. Wescott, F. J. Weyl, Hermann Weyl,

G. W. Whitehead, A. L. Whiteman, E. A. Whitman, P. M. Whitman, W. F. Whitmore, G. T. Whyburn, D. V. Widder, Norbert Wiener, L. R. Wilcox, R. L. Wilder, S. S. Wilks, E. W. Wilson, František Wolf, F. E. Wood, E. R. Worthington, C. H. Yeaton, J. W. T. Youngs, Oscar Zariski, J. W. Zimmer, Antoni Zygmund.

The Colloquium Lectures on the subject *Mathematical relations and structures* were given by Professor Oystein Ore of Yale University Wednesday, Thursday and Friday mornings. The presiding officers were Professors A. A. Albert, C. G. Latimer, and Emil Artin, respectively.

The sixteenth Josiah Willard Gibbs Lecture was given by Professor Sewall Wright of the University of Chicago on Wednesday evening. His title was *Statistical genetics and evolution*. President Marston Morse presided at this lecture.

Four general lectures were given under the composite title *Trends in research*. These were: *Abstracts spaces* by Professor M. H. Stone, Tuesday morning; *Topology* by Professor Solomon Lefschetz, Tuesday afternoon; *The calculus of variations* by Professor G. A. Bliss, Wednesday afternoon; and *The analytic theory of numbers* by Professor H. A. Rademacher, Friday afternoon. The presiding officers for these lectures were, in order, President Marston Morse, and Professors R. L. Wilder, R. W. Brink, and L. E. Dickson.

To present and discuss the latest developments in the fields of Algebra and the Theory of Integration, Conferences were held in these two subjects. These met simultaneously on Wednesday, Thursday and Saturday mornings. In the Conference on Algebra the speakers with their subjects were: Professor Garrett Birkhoff, *Lattice-ordered abelian groups*; Professor John von Neumann, *Complete lattice abelian groups*; Professor N. H. McCoy, *Some aspects of the theory of matrices over a ring*; Professor Nathan Jacobson, *Lie algebras*; Professor J. F. Ritt, *Differential polynomials*; Professor Oscar Zariski, *Some recent results in the arithmetical theory of algebraic varieties*; and Dr. O. F. G. Schilling, *A Hilbert series theory for general algebraic extensions*. The presiding officers of the Conference on Algebra were, in order, Professors A. A. Albert, Richard Brauer, and Hermann Weyl. In the Conference on the Theory of Integration the lectures were: Professor Salomon Bochner, *Integration and partial ordering*; Professor G. B. Price, *Integration and convex operators*; Professor T. H. Hildebrandt, *The general integrals of E. H. Moore*; Professor R. L. Jeffery, *Some recent advances in the theory of non-absolutely convergent integrals*; Professor Norbert Wiener, *Measure in statistical mechanics*; Professor Garrett Birkhoff, *Product integrals*; and Professor Nelson Dunford, *Applications of the theory of integration to the study of linear operators*. The

presiding officers for the three sessions of the Conference on the Theory of Integration were Professors L. M. Graves, John von Neumann, and E. J. McShane.

Thirteen sectional sessions for the reading of short papers were held as follows: Mathematical Statistics (joint session with the Institute of Mathematical Statistics and Econometric Society), Tuesday morning; Analysis, Tuesday morning, Tuesday afternoon (two sections), Friday morning, Friday afternoon; Geometry, Tuesday morning; Topology and the Theory of Sets, Tuesday morning; Algebra, Tuesday afternoon, Friday afternoon; Algebra and the Theory of Numbers, Friday morning; Topology, Friday morning; Applied Mathematics, Friday afternoon. The presiding officers for these sectional sessions were, in order, Professors A. T. Craig, D. R. Curtiss, Dunham Jackson, Salomon Bochner, C. R. Adams, T. H. Hildebrandt, Virgil Snyder, J. W. Tukey, A. A. Bennett, Saunders MacLane, M. H. Ingraham, Solomon Lefschetz, and Norbert Wiener.

Headquarters for the meeting were in Judson Court. The dormitories of the University of Chicago were available to the members of the four mathematical organizations and their guests. The general lectures and the Gibbs Lecture were held in Kent Chemical Laboratory. All other sessions met in Eckhart Hall and the adjoining Ryerson Physical Laboratory.

On Monday evening the members of the Department of Mathematics of the University of Chicago gave an informal reception in Burton Court Lounge. Tea was served in Eckhart Hall on Tuesday and Wednesday afternoons by the ladies of the Department. There was a bridge tournament on Tuesday evening. A luncheon and fashion show were held at Marshall Field and Company on Friday, and Friday evening a visit was made to Station WGN. Thursday afternoon was left free for recreation.

On Thursday evening there was a brief reception with officers of the four organizations receiving. This was followed by the joint dinner in Hutchinson Commons with an attendance of three hundred forty-one. President Marston Morse presided at the dinner. Professor Arnold Dresden spoke of the founding and early days of mathematics at the University of Chicago and particularly of the triumvirate Moore-Bolza-Maschke. At the close of his address he introduced the following resolution of greeting to Professor Oskar Bolza, the sole surviving member of this trio, which was adopted by a unanimous vote:

A large group of members of the American Mathematical Society and their guests, meeting in Chicago to take part in the Celebra-

tion of the Fiftieth Anniversary of the Founding of the University of Chicago, send you their most affectionate greetings, mindful of the great contribution you have made to the development of mathematics in America and to its growth at the University of Chicago.

Professor Dresden was followed by Professor S. S. Wilks, who told of the history and purpose of the Institute of Mathematical Statistics and the Econometric Society, Professor D. R. Curtiss, who spoke of student days in Paris and the French professors of this era, and Professor G. A. Bliss, who read an announcement of the National Science Fund.

At the close of the dinner a resolution, introduced by Professor F. L. Griffin, extending felicitations to the University of Chicago upon its semicentennial and thanking it for the privilege of meeting as a part of the Celebration and expressing thanks to the Department of Mathematics for its fine arrangements and gracious hospitality, was adopted by a unanimous rising vote.

There were two meetings of the Council, one being a joint session with the Board of Governors of the Mathematical Association of America. This joint session to hear the report of the War Preparedness Committee was held at 4:30 P.M. in Eckhart Hall on Monday. This report appears elsewhere in this issue of the Bulletin. The other meeting of the Council was held at 8:15 P.M. on Tuesday in Judson Court Library.

The Secretary announced the election of the following twenty-nine persons to membership in the Society:

Dr. Fred Assadourian, New York University;
Dr. Herman Russell Branson, Dillard University, New Orleans, La.;
Dr. George William Brown, R. H. Macy and Company, New York, N. Y.;
Mr. George Eaddie Bulloch, New Jersey Bell Telephone Company, Newark, N. J.;
Professor Elías Alfredo de Cesare, University of La Plata;
Dr. Myron Abraham Coler, Paragon Paint and Varnish Corporation, Long Island City, N. Y.;
Mr. Alexander Robert Craw, University of Notre Dame;
Dr. Joseph Francis Daly, Catholic University of America;
Professor Max Wilhelm Dehn, University of Idaho;
Mr. Jacques Dutka, Bronx, New York, N. Y.;
Mr. E. LaVerne Eagle, Dayton Y.M.C.A. College, Dayton, Ohio;
Professor Laurence Hadley, Purdue University;
Professor Louise Stokes Hunter, Virginia State College;
Professor Alfred Edward Johns, McMaster University, Hamilton, Ontario, Canada;
Mr. Seymour Matthew Kwerel, Columbia University;
Mr. Thorstein Larsen, Consolidated Edison Company, Brooklyn, N. Y.;
Dr. Charles Loewner, University of Louisville;

Dr. Robert Sewell Pate, University of Illinois;
Mr. Murray Harold Protter, Brown University;
Professor Frank Atkinson Rickey, Louisiana State University;
Professor Mario O. González Rodríguez, University of Havana;
Dr. Leonard Jimmie Savage, Institute for Advanced Study;
Dr. William John Schart, Ohio State University;
Dr. Joseph A. Sharpe, Stanolind Oil and Gas Company, Tulsa, Okla.;
Dr. Ronald William Shephard, University of California;
Mr. William Henry Smith, Jr., Washington, D. C.;
Mr. Jonathan Dean Swift, California Institute of Technology;
Dr. Leopoldo Toralballa, Government Service Insurance System, Manila, Philippine Islands;
Dr. František Wolf, Macalester College, St. Paul, Minn.

It was reported that President Marston Morse will represent the Society at the One Hundred and Seventy-fifth Anniversary of the Founding of Rutgers University on October 9, 10, and 11, 1941. The following appointment by President Morse was also reported: as a subcommittee of the War Preparedness Committee, to be known as the Committee on Supply and Demand for Mathematicians, Professors T. Y. Thomas (chairman), Harry Bateman, E. T. Bell, G. C. Evans, and W. M. Whyburn.

Reports were sent in August to the Carnegie Corporation and Rockefeller Foundation regarding their grants to Mathematical Reviews.

The Secretary reported that copies of the recent report entitled *Industrial Mathematics* by Dr. T. C. Fry were distributed in August to members of the Society who are not members of the Mathematical Association of America.

Certain invitations to give hour addresses were announced: Professor G. B. Price for the November 1941 meeting in Manhattan, Kansas; Professors L. V. Bewley, I. S. Sokolnikoff and Oscar Zariski for the 1941 Annual Meeting at Lehigh University; Professor Norman Levinson for the February 1942 meeting in New York; Professor L. M. Blumenthal for the April 1942 meeting in Chicago; and Professor W. L. Ayres for the 1942 Summer Meeting at Cornell University.

On recommendation of the Committee on Places of Meetings (Professor A. B. Coble, chairman), the eastern spring meeting of 1942 was set at New York City, April 3-4. Times and places of meetings during 1942 were also set as follows: in New York City on February 28; in Berkeley, California on April 11; in Chicago on April 17-18; at Cornell University on September 8-11; in New York City on October 31; at the University of Notre Dame on November 27-28.

The seventeenth Josiah Willard Gibbs Lecture is to be given by

Professor John von Neumann at the Annual Meeting of 1942 in New York City.

Dean L. P. Eisenhart and Dean R. G. D. Richardson were appointed representatives on the Council of the American Association for the Advancement of Science for 1942 and Professor M. H. Stone was appointed representative on the National Research Council for a period of three years beginning July 1942.

The committee on the proposed new series of monographs, *Mathematical Surveys* (Professor J. L. Walsh, chairman), was empowered to proceed with the proposal presented by the Williams and Wilkins Company. In accordance with this proposal, the Company has agreed to publish at least two monographs, without commitment by the Society or the Company as to continuation. This committee was also empowered to appoint a temporary editorial board to act in connection with arrangements for the two monographs.

Professor G. B. Price reported for the Committee on Circulation for Mathematical Reviews that the total mailing list had increased from 1,222 on July 1, 1940 to 1,369 on July 1, 1941. He also stated that new subscriptions are continuing to arrive from China, Japan, Russia, England, et cetera, thus emphasizing the truly international character of this journal.

A committee consisting of Professors M. H. Ingraham (chairman), L. M. Graves, and J. W. Tukey was appointed to consider the question of dues for persons drafted or enlisted for military service.

The Council voted to recommend to the Society that Section 9, Article VII of the By-Laws be amended so that no new life members will be accepted. Action will be taken on this proposal at the October meeting in New York City.

Titles and cross references to the abstracts of the papers read at this meeting follow below. Papers numbered 1 to 6 were read before the section for Mathematical Statistics, papers 7 to 12 before the section for Analysis Tuesday morning, papers 13 to 19 before the section for Geometry, papers 20 to 24 before the section for Topology and the Theory of Sets, papers 25 to 28 before the first section for Analysis Tuesday afternoon, papers 29 to 33 before the section for Algebra Tuesday afternoon, papers 34 to 38 before the second section for Analysis Tuesday afternoon, papers 39 to 45 before the section for Analysis Friday morning, papers 46 to 51 before the section for Algebra and the Theory of Numbers, papers 52 to 59 before the section for Topology, papers 60 to 66 before the section for Analysis Friday afternoon, papers 67 to 72 before the section for Algebra Friday afternoon, and papers 73 to 76 before the section for Applied

Mathematics. Papers 77 to 141, whose abstract numbers are followed by the letter *i*, were read by title. Paper number 9 was read by Professor Schaeffer, 10 by Dr. Bankier, 11 by Mr. Thron, 29 by Dr. Pate, 40 by Dr. Duffin, 58 by Professor Eilenberg, 60 by Dr. Kac, 61 by Dr. Halmos, and 62 by Professor Kakutani. Professor Burr was introduced by Professor W. L. Ayres, Dr. Mood by Professor E. G. Olds, Mr. Allen by Dr. P. E. Lewis, Mr. Young by Professor R. L. Moore, Professor Dehn by Professor Oswald Veblen, Dr. Weinstein by Dr. H. E. Robbins, and Dr. Nicholson by Professor S. E. Warschawski.

1. J. B. Coleman: *A geometric derivation of Fisher's z-transformation*. (Abstract 47-9-433.)
2. Abraham Wald: *Large sample distribution of the likelihood ratio*. (Abstract 47-11-486.)
3. I. W. Burr: *Cumulative frequency functions*. (Abstract 47-9-432.)
4. K. J. Arnold: *On spherical probability distributions*. (Abstract 47-9-431.)
5. A. M. Mood: *On the asymptotic distribution of medians of samples from a multivariate population*. (Abstract 47-9-436.)
6. J. F. Daly: *A problem in estimation*. (Abstract 47-11-483.)
7. L. R. Ford: *The proper fractions*. (Abstract 47-9-379.)
8. Otto Szász: *Some new summability methods with applications*. (Abstract 47-9-404.)
9. R. J. Duffin and A. C. Schaeffer: *Power series with bounded coefficients*. (Abstract 47-9-376.)
10. J. D. Bankier and Walter Leighton: *Arithmetic continued fractions*. (Abstract 47-9-366.)
11. Walter Leighton and W. J. Thron: *Convergence criteria for continued fractions*. (Abstract 47-7-308.)
12. J. F. Paydon: *Convergence regions and value regions for continued fractions*. Preliminary report. (Abstract 47-11-473.)
13. Abraham Schwartz: *A consequence of the Ricci equations for riemannian manifolds*. (Abstract 47-9-426.)
14. T. R. Holcroft: *Types of distinct double points of primals*. (Abstract 47-9-423.)
15. V. N. Robinson: *A transformation between osculating curves of a rational normal curve in an odd dimensional space*. (Abstract 47-7-323.)
16. J. J. DeCicco: *Geometric characterization of functions of n complex variables*. (Abstract 47-9-420.)
17. F. E. Hohn: *On Cartan's projective involutes*. (Abstract 47-9-422.)

18. J. H. Giese: *Conformally flat hypersurfaces of spaces of constant curvature*. (Abstract 47-9-421.)
19. E. F. Allen: *On a triangle inscribed in a rectangular hyperbola*. (Abstract 47-9-417.)
20. R. G. Lubben: *Decompositions of point sets and of portions of spaces*. (Abstract 47-9-448.)
21. Horace Komm: *Partial orders in euclidean spaces*. (Abstract 47-9-446.)
22. S. M. Ulam: *Theory of the operation of products of sets*. I. Preliminary report. (Abstract 47-9-406.)
23. F. A. Ficken: *Closure-mappings of partially ordered sets*. Preliminary report. (Abstract 47-9-441.)
24. Henry Blumberg: *A reconsideration of the paradoxes in the theory of sets*. (Abstract 47-9-429.)
25. R. H. Cameron: *Quadratic convolution equations*. (Abstract 47-9-370.)
26. R. E. Langer: *A theory for ordinary linear differential boundary problems of highly irregular type*. (Abstract 47-9-391.)
27. I. E. Perlin: *A calculus of variations problem with end points as functions of the curve*. Preliminary report. (Abstract 47-9-398.)
28. František Wolf: *On generalized boundary values and the uniqueness of corresponding Dirichlet problems*. (Abstract 47-9-407.)
29. R. J. Duffin and R. S. Pate: *Generalized groups and the Jordan-Hölder theorem*. (Abstract 47-9-340.)
30. Morris Marden: *The zeros of certain composite polynomials*. (Abstract 47-9-345.)
31. H. H. Campaigne: *An example of a cogroup*. (Abstract 47-9-337.)
32. D. C. Murdoch: *Automorphisms and quasi-groups*. Preliminary report. (Abstract 47-9-347.)
33. Rufus Oldenburger: *The factorization of polynomials*. (Abstract 47-9-349.)
34. D. G. Bourgin: *Some properties of Banach spaces*. (Abstract 47-9-369.)
35. F. B. Jones: *Measure and other properties of a Hamel basis*. (Abstract 47-9-386.)
36. R. S. Phillips: *On weakly compact subsets of a Banach space*. Preliminary report. (Abstract 47-9-399.)
37. M. M. Day: *Reflexivity criteria for a Banach space*. (Abstract 47-9-374.)
38. Herman Meyer: *Polynomial approximations to functions defined on abstract spaces*. (Abstract 47-9-395.)

39. Bernard Friedman: *Fourier coefficients of odd functions*. (Abstract 47-9-380.)
40. R. J. Duffin and J. J. Eachus: *A Paley-Wiener type expansion theorem*. (Abstract 47-9-375.)
41. R. P. Boas: *Generalized Laplace integrals*. (Abstract 47-7-303.)
42. R. P. Agnew: *Analytic extension by Hausdorff methods*. (Abstract 47-9-363.)
43. J. H. Curtiss: *Riemann sums and the fundamental polynomials of Lagrange interpolation*. (Abstract 47-7-305.)
44. J. M. Earl: *Polynomials of interpolation on an infinite interval*. (Abstract 47-9-377.)
45. R. L. Moore: *On continua with dendratomic subsets*. (Abstract 47-11-488.)
46. J. M. Dobbie: *A certain method in additive number theory*. Preliminary report. (Abstract 47-9-360.)
47. R. D. James: *On the sieve method of Viggo Brun*. (Abstract 47-9-361.)
48. A. T. Brauer: *On the least quadratic non-residue*. (Abstract 47-9-357.)
49. Albert Whiteman: *Sums connected with the partition function*. (Abstract 47-7-300.)
50. Fritz Herzog: *On an application of a theorem by Burnside*. (Abstract 47-9-341.)
51. Gordon Pall: *An elementary alternative to Dirichlet's theorem*. (Abstract 47-11-465.)
52. G. S. Young: *On topologically flat subsets of certain spaces of R. L. Moore*. (Abstract 47-9-452.)
53. O. G. Harrold: *A mapping characterization of Peano spaces*. (Abstract 47-9-444.)
54. R. H. Fox: *Homotopy type and deformation retraction*. (Abstract 47-9-442.)
55. N. E. Steenrod: *Topological methods for the construction of tensor functions*. (Abstract 47-11-489.)
56. Max Dehn: *On the mapping of closed surfaces*. (Abstract 47-9-440.)
57. A. W. Tucker: *A boundary-value theorem for harmonic tensors*. (Abstract 47-9-449.)
58. Samuel Eilenberg and Saunders MacLane: *Infinite cycles and homologies*. (Abstract 47-11-487.)
59. Henry Wallman: *Dimension for general spaces*. (Abstract 47-11-490.)



60. R. P. Agnew and Mark Kac: *On differences of elements of sets having positive Haar measure.* (Abstract 47-9-365.)

61. P. R. Halmos and John von Neumann: *Operator methods in classical mechanics.* II. (Abstract 47-9-385.)

62. Shizuo Kakutani and Warren Ambrose: *Structure and continuity of measurable flows* (Abstract 47-9-389.)

63. Szolem Mandelbrojt: *New proof of the conditions of quasi-analyticity.* (Abstract 47-9-393.)

64. J. W. T. Youngs: *A generalized Lebesgue integral.* (Abstract 47-9-409.)

65. Arthur Rosenthal: *On differentiation of integrals and approximate continuity.* (Abstract 47-9-401.)

66. R. B. Kershner: *On the packing of convex regions in the plane.* Preliminary report. (Abstract 47-9-390.)

67. A. D. Campbell: *On the application of barycentric coordinates to lattice theory.* (Abstract 47-9-338.)

68. A. A. Albert: *Non-associative algebras.* I. (Abstract 47-9-331.)

69. L. R. Wilcox: *Complementation and modularity in lattices.* (Abstract 47-9-356.)

70. Richard Brauer: *On the indecomposable representations of algebras.* (Abstract 47-9-334.)

71. R. M. Thrall: *On symmetrized Kronecker powers and Lie representations of the full linear group.* (Abstract 47-9-353.)

72. G. K. Kalisch: *On Jordan algebras.* (Abstract 47-9-344.)

73. Wilfred Kaplan: *Topology of the two-body problem.* (Abstract 47-9-412.)

74. Eric Reissner: *On the equations of the stability theory of thin shells.* (Abstract 47-9-415.)

75. Isaac Opatowski: *On the theory of lethal irradiation of microorganisms.* I. (Abstract 47-9-414.)

76. A. S. Householder: *A theory of steady-state activity in nerve-fiber networks.* IV: n circuits with a common synapse. (Abstract 47-9-411.)

77. W. K. Feller: *On the integral equation of the renewal theory.* (Abstract 47-9-435-t.)

78. Hilda P. Geiringer: *Some observations on analysis of variance theory.* (Abstract 47-11-484-t.)

79. Raphaël Salem: *On sets of multiplicity for trigonometrical series.* (Abstract 47-9-402-t.)

80. T. F. Glass and Walter Leighton: *On the convergence of continued fractions.* (Abstract 47-9-384-t.)

81. J. P. LaSalle: *A characterization of topological spaces.* (Abstract 47-9-447-t.)
82. A. D. Wallace: *Regularly ordered systems of sets.* (Abstract 47-9-450-t.)
83. Ralph Mansfield: *Differential systems involving k -point boundary conditions.* (Abstract 47-9-394-t.)
84. H. A. Simmons: *Classes of maximum numbers associated with two symmetric equations in n reciprocals.* (Abstract 47-9-351-t.)
85. Stefan Bergman and D. C. Spencer: *Some properties of pseudo-conformal transformations in the neighborhood of boundary points.* (Abstract 47-9-368-t.)
86. B. W. Jones: *On an extension of a theorem of Witt.* (Abstract 47-9-343-t.)
87. Alexander Weinstein: *On the buckling of a rectangular clamped plate compressed in one direction.* (Abstract 47-9-416-t.)
88. Edward Kasner and J. J. DeCicco: *Geometry of dual-velocity systems.* (Abstract 47-5-265-t.)
89. Edward Kasner and J. J. DeCicco: *General differential geometry of second order differential elements.* (Abstract 47-7-321-t.)
90. Edward Kasner and J. J. DeCicco: *Conformal geometry of series of third order differential elements.* (Abstract 47-7-320-t.)
91. Edward Kasner and J. J. DeCicco: *Pseudo-conformal geometry: functions of two complex variables.* (Abstract 47-9-424-t.)
92. J. J. DeCicco: *Equilong geometry of series of collinear third order differential elements.* (Abstract 47-7-319-t.)
93. E. N. Oberg: *Notes on the approximation of a function by sums of orthonormal functions.* Preliminary report. (Abstract 47-7-311-t.)
94. T. A. Botts: *On convex sets in linear normed spaces.* (Abstract 47-7-327-t.)
95. J. D. Mancill: *On the solutions of a certain class of partial differential equations.* (Abstract 47-7-310-t.)
96. A. W. Jones: *On the characterization of groups whose lattices satisfy specified lattice identities.* (Abstract 47-7-295-t.)
97. C. J. Everett: *Vector spaces over rings.* (Abstract 47-7-294-t.)
98. Nathaniel Coburn: *Semi-analytic unitary subspaces of unitary spaces.* (Abstract 47-9-419-t.)
99. Gordon Pall: *Quaternions and sums of three squares.* (Abstract 47-9-350-t.)
100. P. W. Carruth: *Arithmetic of ordinals with applications to the theory of ordered abelian groups.* (Abstract 47-9-339-t.)
101. Marston Morse and G. A. Hedlund: *Manifolds without conjugate points.* (Abstract 47-9-425-t.)

102. E. A. Knoblauch: *Extension of homeomorphisms on the torus and projective plane*. (Abstract 47-9-445-t.)
103. J. D. Mancill: *Multiple integral problems of the calculus of variations with prescribed transversality coefficients*. (Abstract 47-9-392-t.)
104. Louis Weisner: *Roots of certain classes of polynomials*. (Abstract 47-9-355-t.)
105. František Wolf: *On the summability of trigonometrical integrals*. (Abstract 47-9-408-t.)
106. Orrin Frink: *Topology in lattices*. (Abstract 47-9-443-t.)
107. M. M. Day: *Ergodic theorems for abelian semi-groups*. (Abstract 47-9-372-t.)
108. M. M. Day: *Operations in Banach spaces*. (Abstract 47-9-373-t.)
109. H. L. Garabedian: *Theorems relating to the Cesàro kernel transformation*. (Abstract 47-9-382-t.)
110. H. L. Garabedian: *A class of linear integral transformations*. (Abstract 47-9-381-t.)
111. Deane Montgomery and Leo Zippin: *A theorem on Lie groups*. (Abstract 47-9-346-t.)
112. Garrett Birkhoff: *Metric foundations of geometry*. (Abstract 47-9-428-t.)
113. Olga Taussky and Ernest Best: *A class of groups*. (Abstract 47-9-352-t.)
114. A. A. Albert: *Division algebras over a function field*. (Abstract 47-9-330-t.)
115. D. C. Spencer: *A function-theoretic identity*. (Abstract 47-9-403-t.)
116. C. C. Camp: *A convergence proof involving an inseparable multiple contour integral*. (Abstract 47-9-371-t.)
117. S. A. Jennings: *Central chains of ideals in an associative ring*. (Abstract 47-9-342-t.)
118. R. P. Agnew: *On Hurwitz-Silverman-Hausdorff methods of summability*. (Abstract 47-9-364-t.)
119. R. H. Bruck and T. L. Wade: *On a tensor algebra of Cramlet*. (Abstract 47-9-336-t.)
120. T. L. Wade: *Euclidean concomitants of the ternary cubic*. (Abstract 47-9-354-t.)
121. R. H. Bruck: *The number of absolute invariants of a tensor*. (Abstract 47-9-335-t.)
122. Mark Kac: *Convergence and divergence of non-harmonic gap series*. (Abstract 47-9-387-t.)

123. Shizuo Kakutani: *A class of examples of mixing flows*. (Abstract 47-9-388-t.)
124. E. L. Buell: *On the distribution of plane stress in a semi-infinite plate with partially stiffened edge*. Preliminary report. (Abstract 47-9-410-t.)
125. G. S. Young: *On the outer boundaries of certain connected domains*. (Abstract 47-9-451-t.)
126. Abe Gelbart: *On some properties of mapping functions*. (Abstract 47-9-383-t.)
127. M. G. Moore: *On functions satisfying two functional equations*. (Abstract 47-9-396-t.)
128. Richard Brauer: *Investigations on group characters*. (Abstract 47-9-333-t.)
129. Brockway McMillan: *Absolutely monotone functions*. Preliminary report. (Abstract 47-11-471-t.)
130. Ivan Niven: *Quadratic diophantine equations in the rational and quadratic fields*. (Abstract 47-9-362-t.)
131. Ivan Niven: *The m th roots of a quaternion*. (Abstract 47-9-348-t.)
132. L. R. Wilcox: *Extensions of semi-modular lattices*. II. (Abstract 47-11-461-t.)
133. Leonard Carlitz: *Note on generalized Bernoulli and Euler numbers*. (Abstract 47-9-358-t.)
134. Leonard Carlitz: *The coefficients of the reciprocal of a series*. (Abstract 47-9-359-t.)
135. E. H. Nicholson: *On the degree of approximation in some convergence theorems concerning derivatives of the mapping function in conformal mapping*. (Abstract 47-9-397-t.)
136. S. M. Ulam and D. H. Hyers: *On approximate isometries*. Preliminary report. (Abstract 47-9-427-t.)
137. S. M. Ulam: *On measures for subsets of sets of measure zero*. (Abstract 47-9-405-t.)
138. E. F. Beckenbach: *Vector formulations of Morera's theorem*. (Abstract 47-9-367-t.)
139. Paul Erdős and Gabor Szegő: *On a problem of I. Schur*. (Abstract 47-9-378-t.)
140. Reinhold Baer: *Automorphism rings of primary abelian operator groups*. (Abstract 47-9-332-t.)
141. H. S. Vandiver: *An arithmetical theory of Bernoulli numbers*. (Abstract 47-11-466-t.)

W. L. AYRES,
Associate Secretary

BOOK REVIEWS

Mathematics Applied to Electrical Engineering. By A. G. Warren. New York, Van Nostrand, 1940. 384 pp., 131 figs. \$4.50.

This is a review course in college mathematics for electrical engineers. Some pages read as if they were a part of a mathematical text in electricity and magnetism, except that the emphasis is on mathematics; other pages read as if they were out of a book on "unified" college mathematics; in its entirety the book is a compact presentation of a number of useful mathematical principles and methods, illustrated by a wide selection of electrical problems.

Thus we find: complex numbers and their use in steady state electric circuit problems; linear differential equations with constant coefficients and their application to transients; the Bessel equation and the skin effect problem; partial differential equations and a discussion of plane waves, spherical waves and waves along wires; elements of functions of complex variable and problems of steady current flow; Heaviside's operational calculus and transients in transmission lines.

The above is not a complete list of either the kinds of mathematics or types of problems (some more elementary than others), which are taken up in the book. To help those who may be "rusty" on the principles of electricity and magnetism the author provides brief review chapters. Altogether, the book may appeal to college students and to practicing engineers—to the latter as a review book in fundamentals and to the former as a bridge between their mathematical and engineering studies.

SERGEI A. SCHELKUNOFF

Elemente der Operatorenrechnung mit geophysikalischen Anwendungen.

By Hans Ertel. Berlin, Springer, 1940. 6+133 pp. RM 7.50.

This monograph demonstrates the applications of the operational calculus according to Heaviside to problems in geophysics. In the first section some pertinent remarks are made about certain simple differential equations with the view of later applications. The second section is concerned with the derivation and a brief physical discussion of some of the partial differential equations of mathematical physics, in particular the Stokes-Navier equations, the Maxwell equations and other equations derived from these. The third and fourth sections are concerned with a discussion of the Heaviside operational calculus and its applications to a variety of interesting problems in geophysics.

This text gives a brief but excellent account of the methods of mathematical physics. Despite the fact that this monograph is concerned primarily with the applications of the operational calculus to geophysics, one interested in a bird's-eye view of certain classical phases of mathematical physics can find them put down in short order. Many unusual problems of theoretical physics with geophysical applications are to be found.

A. E. HEINS

Matrix and Tensor Algebra for Engineers and Chemists. By Clarence E. Rose. Brooklyn, Chemical Publishing Co., 1940. 8+143 pp. \$4.00.

Matrix and Tensor Algebra has been written "for electrical and communication engineers and chemists who wish to learn about determinants, vectors and tensors." Its only mathematical interest lies in the complete novelty of its point of view.

For example, no need is felt for proofs; theorems are stated as *obiter dicta*. Such definitions as "a matrix is a group of independent but related quantities," "a determinant is a finite, imaginary, or complex quantity, or generalization of the algebraic number zero" illustrate its unorthodoxy further. However, as the book was not designed for mathematicians, it would be out of place to go into more detail.

GARRETT BIRKHOFF

Compendio di Meccanica Razionale. 2d revised edition. By Tullio Levi-Civita and Ugo Amaldi. Bologna, Zanichelli, 1938. 12+423+8+310 pp.

The two thousand page work by the same authors, entitled *Lezioni di Meccanica Razionale*, has already been adequately reviewed by K. P. Williams (this Bulletin, vol. 36, p. 781). It is one of the great standard works on mechanics of almost an encyclopaedic nature.

The *Compendio*, the work now under review, represents an attempt to cut down the *Lezioni* to the usual academic course of study in Italian universities and technical schools. This has been done by wholesale omissions of entire subjects rather than by abridged exposition. On the contrary, the emphasis on the foundations of the subject has been strengthened by a certain amount of additional material. Also included are a few minor corrections and revisions. For the most part, however, whole sections of the *Compendio* read word for word exactly as do the corresponding sections in the *Lezioni*.

Some of the subjects omitted in the *Compendio* are rather sur-

prising. We find, for example, no mention of the principle of least action, of Hamilton's canonical equations, or a fortiori of the Hamilton-Jacobi partial differential equation, or of invariant integrals. Thus there is necessarily no account of much of the important progress made during the last fifty years on the nature of conservative holonomic dynamical systems.

The length of the *Compendio* is further reduced by the omission of all exercises; for these the authors recommend the use of a book by Bisconcini (*Esercizi e Complementi di Meccanica Razionale*).

With all these omissions the work still runs to over seven hundred pages and thus still partakes something of an encyclopaedic nature in the more elementary parts of the subject. The balance between the statics and kinematics of the first part and the dynamics of the second part is, in our opinion, unfortunate. We should prefer a smaller first part and a larger second, especially inasmuch as the second part also contains a small amount of hydrodynamics and elasticity theory. For this reason we doubt the feasibility of using this work as a textbook in an American university, even if it were translated into English. But as a clearly and charmingly written reference book it might well prove invaluable.

D. C. LEWIS

Barlow's Tables of Squares, Cubes, Square Roots, Cube Roots and Reciprocals of all Integer Numbers up to 12,500. Edited by L. J. Comrie. 4th edition. London, Spon; Brooklyn, Chemical Publishing Co., 1941. 12+258 pp. \$3.00.

The book is accurately described by its title. Roots are given to six places of decimals, and reciprocals to seven significant figures. The directions given for interpolation and for computing additional figures are adapted to the use of computing machines. There is appended a short table of powers of integers from 1 to 100 up to the tenth, and of those from 1 to 10 up to the twentieth.

Barlow's Tables are of historical interest. First prepared and published in 1814 by Peter Barlow of the Royal Military Academy (Woolwich), they were edited and reissued by A. De Morgan in 1839. The plates ultimately wore out and a third edition was prepared in 1930 by Dr. Comrie of H. M. Nautical Almanac Office. The present is a revision of the third edition, in which the extension from 10,000 to 12,500 is new. The prefaces to the first three editions are included in the present text.

R. A. JOHNSON

Introduction to Algebraic Theories. By A. Adrian Albert. The University of Chicago Press, 1941. 8+137 pp. \$1.75.

Those who have learned abstract algebra know that its methods are at the same time simpler, easier to understand and more penetrating than the more classical procedures; but those who have taught abstract algebra know that there is one great obstacle in the way of the student desirous of acquainting himself with this new approach to algebra. The novice has first to learn to work with concepts, to "compute" with them as easily as he used to do with numbers and such like; though after having succeeded in this, he finds that everything else is comparatively easy going.

It is the object of this *Introduction to Algebraic Theories* to help the student in his attempt to overcome the difficulties just indicated and to acquire those habits of abstract thinking which are the indispensable foundation not only of abstract algebra, but of almost all present day mathematics. Since this book is intended to be a book of preparation, it does not require any previous knowledge—apart from algebraic techniques already known to the mathematicians of the Renaissance—though a willingness to work and think are indispensable.

The topics treated in this text are those which one would naturally expect: matrices, their equivalence and similarity, as well as a number of related subjects. This entails some study of polynomials—for technical reasons—and of linear spaces, since the latter are indispensable for understanding the real significance of the concepts pertaining to matrices. Indications of the technique of generalization—so important today—and of its usefulness are given throughout. The culmination of the whole work may be seen in the last chapter in which the fundamental concepts of abstract algebra are introduced as the natural outcome of the preceding considerations.

A great number of exercises—both numerical illustrations and mathematical applications—provide the reader with an opportunity to test the acquired skills. The book is written with the clarity of style, the arguments are presented with the elegance and precision which one has learned to expect from its author. Thus this text will prove valuable to student and teacher alike.

REINHOLD BAER

NOTES

Copies of the report *Mathematical Education for Defense* of the Subcommittee on Education for Service of the War Preparedness Committee (W. L. Hart, Chairman) may be obtained, while the supply lasts, in lots of 25 for \$1.25 from Professor J. R. Kline, University of Pennsylvania, Philadelphia, Pa., or Professor W. D. Cairns, 97 Elm Street, Oberlin, Ohio. This report is reprinted from the American Mathematical Monthly and has also appeared in the Mathematics Teacher and School Science and Mathematics.

During the winter 1941-1942 a symposium on non-linear circuit theory will be presented by the Basic Science Group of the American Institute of Electrical Engineers, New York Section. The following six lectures will be given: *General nature of the problem of non-linear circuits and outline of the various methods of attack*, Professor E. Weber, Polytechnic Institute of Brooklyn; *Power series solutions*, Dr. L. A. MacColl, Bell Telephone Laboratories; *Fourier series solutions*, Mr. W. R. Bennett, Bell Telephone Laboratories; *Graphical methods*, Mr. A. Preisman, R.C.A. Institute; *Numerical step-by-step method*, Professor Alan Hazeltine, Stevens Institute of Technology; *Analytical methods*, Dr. P. Le Corbeiller, Harvard University and New York University. Further information about this symposium may be obtained from Mr. H. E. Farrer, A.I.E.E. Headquarters, 33 West 39th Street, New York, N. Y.

Applications for Benjamin Peirce instructorships at Harvard University for the year 1942-1943 should be sent to the Chairman of the Department of Mathematics. Candidates should have received the Ph.D. degree or have had equivalent training.

The School of Mathematics of the Institute for Advanced Study each year allocates a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics of the Institute, Fuld Hall, Princeton, N. J., and are returnable by February 1, 1942.

The first Charles Chree Medal and Prize of the English Physical Society has been awarded to Professor Sydney Chapman of the Imperial College of Science and Technology.

According to an announcement in *Mathematische Annalen* (vol.

117, no. 5, p. 776) Jablonowskischen Gesellschaft der Wissenschaften is offering a prize of 500 RM for an investigation of older Greek arithmetic and algebra. Manuscripts must be in German or Latin and will be received up to the end of 1942.

Professor G. D. Birkhoff of Harvard University has been elected an honorary member of the London Mathematical Society.

An honorary doctorate of science has been conferred by Washburn College upon Professor L. M. Graves of the University of Chicago.

The honorary degree of doctor of laws has been conferred upon Dean R. G. D. Richardson of Brown University by Lehigh University.

Guggenheim Fellowships for 1941-1942 have been awarded to Professor Carrillo Flores of the National University of Mexico, Assistant Professor Richard Brauer of the University of Toronto, Associate Professor Deane Montgomery of Smith College, and Dr. Jesse Douglas.

Professor David van Dantzig of the Institute of Technology, Jaffalaan, Delft, Netherlands, has retired.

The following additional members of the American Mathematical Society are in residence at the Institute for Advanced Study for all or part of this academic year: Dr. Valentin Bargmann, Dr. David Blackwell, Dr. A. T. Brauer, Professor Richard Brauer of the University of Toronto, Dr. Norman Davids, Professor J. L. Doob of the University of Illinois, Dr. Guido Fubini, Dr. Kurt Gödel, Dr. P. R. Halmos, Dr. M. H. Heins, Dr. G. P. Hochschild, Professor Shizuo Kakutani of Osaka Imperial University, Mr. G. K. Kalisch, Mr. G. W. Mackey, Dr. Dorothy Maharam, Professor Deane Montgomery of Smith College, Dr. L. J. Savage, and Dr. R. M. Thrall of the University of Michigan.

The following persons are also to be at the Institute for Advanced Study during a part or all of the current academic year: Dr. Felix Adler, Professor Subrahmanyan Chandrasekhar of Yerkes Observatory, Professor K. S. Cole of Columbia University College of Physicians and Surgeons, Dr. S. M. Dancoff, Professor Wolfgang Pauli of Eidgenössische Technische Hochschule, Zürich, Dr. Hans Samelson, Professor Mario Schenberg of University of São Paulo, Brazil, Professor C. L. Siegel formerly of the University of Göttingen, and Mr. A. H. Stone.

Dr. H. R. Branson of Dillard University has been appointed to an assistant professorship at Howard University.

Associate Professor I. W. Burr of Antioch College has been appointed to an assistant professorship at Purdue University.

Dr. W. B. Caton of Athens College, Athens, Alabama, has been appointed acting head of the mathematics department at Southwestern College, Winfield, Kansas.

Dr. F. G. Dressel of Duke University has been promoted to an assistant professorship.

Professor Emeritus W. F. Durand of Stanford University has been appointed a member of the National Advisory Committee on Aeronautics.

Dr. Samuel Eilenberg of the University of Michigan has been promoted to an assistant professorship.

Professor R. E. Gaines of the University of Richmond has retired as head of the department of mathematics.

Dr. A. S. Galbraith of the University of Rochester has been appointed to an assistant professorship at Colby College.

Assistant Professor F. C. Gentry of Louisiana Polytechnic Institute has been promoted to an associate professorship.

Dr. W. N. Hallett has accepted a position at Frostburg State Teachers College, Frostburg, Maryland.

Professor A. M. Harding of the University of Arkansas has recently been made president of the University.

Professor I. M. Hostetter of Howard College, Birmingham, Alabama, has been appointed to an assistant professorship at Oregon State College.

Dr. R. N. Johanson of Bradley Polytechnic Institute has been promoted to an assistant professorship.

Dr. P. M. Kendig of the Schuylkill Undergraduate Center, Pennsylvania State College, has been appointed to an assistant professorship of physics at Franklin and Marshall College.

President E. O. Lovett of Rice Institute, formerly professor of mathematics and astronomy at Princeton University, has retired at the age of seventy with the title president emeritus.

Assistant Professor E. W. Miller of the University of Michigan has been promoted to an associate professorship.

Dr. C. J. Nesbitt of the University of Michigan has been promoted to an assistant professorship.

Dr. L. F. Ollmann of the College of Wooster has been appointed to an associate professorship at Hofstra College.

Dr. E. D. Rainville of the University of Michigan has been promoted to an assistant professorship.

Assistant Professor G. de B. Robinson of the University of Toronto has an appointment from the National Research Council of Canada.

Dr. R. M. Robinson of the University of California has been promoted to an assistant professorship.

Dr. O. K. Sagen has been appointed chief statistician of the Department of Public Health at Springfield, Illinois.

Dr. W. J. Schart of Ohio State University has been appointed professor of mathematics at the Aviation Cadet Replacement Center, Maxwell Field, Alabama.

Dr. C. E. Sealander of the State University of Iowa has been appointed to an assistant professorship at the University of South Dakota.

Mr. J. W. Sheedy of Michigan State College has been promoted to an assistant professorship.

Associate Professor I. S. Sokolnikoff of the University of Wisconsin has been promoted to a professorship.

Dr. Anna A. Stafford of the University of Utah has been promoted to an assistant professorship.

Associate Professor E. C. Stopher of Ashland College, Ashland, Ohio, has accepted a position at Brockport State Normal School, Brockport, New York.

Mr. E. A. Sturley of Yale University has accepted a position at Berkshire School, Sheffield, Massachusetts.

Dr. G. B. Van Schaack of Michigan State College has been promoted to an assistant professorship.

Dr. W. F. Whitmore of the University of California has accepted a position at the Naval Ordnance Laboratory, Washington, D. C.

Assistant Professor Louise A. Wolf of the University of Wisconsin at Milwaukee has been appointed lecturer at the University of Wisconsin.

Associate Professor R. V. Churchill of the University of Michigan is on leave of absence for the first semester of 1941-1942 and is visiting lecturer at the University of Wisconsin.

Assistant Professor C. H. Fischer of Wayne University is this year visiting assistant professor of mathematics at the University of Michigan.

Assistant Professor A. S. McMaster of the University of Colorado is on leave during the academic year 1941-1942 and is studying at the Iowa State College.

The following appointments to instructorships are announced: University of Chicago: Dr. P. V. Reichelderfer; Colorado College: Dr. Margaret M. Hansman; Cornell University: Dr. Joseph Lehner, Mr. J. C. Smith; Illinois Institute of Technology: Dr. W. S. Snyder; Indiana Technical College: Mr. H. S. Kieval; University of Maryland: Dr. L. C. Hutchinson; Michigan State College: Dr. E. E. Blanche, Mr. Vladimir Morkovin; University of Michigan: Dr. C. J. Thorne; Mundelein College, Chicago, Illinois: Mr. E. L. Godfrey; New Mexico College of Agriculture and Mechanic Arts: Mr. J. A. Joseph; Northwestern University: Dr. S. J. Lawwill, Mr. J. F. Paydon; Ohio State University: Mr. H. D. Huskey, Mr. C. G. Maple; University of Oklahoma: Dr. Ingo Maddaus; Oregon State College: Dr. Rhoda Manning; Pennsylvania State College: Dr. Abraham Schwartz; University of Pennsylvania: Mr. E. K. Ritter; Phillips Academy, Andover, Massachusetts: Mr. W. B. Morgan; Princeton University: Dr. J. H. Giese, Dr. M. S. Macphail, Dr. Henry Scheffé; Purdue University: Dr. G. W. Whitehead, Dr. J. W. T. S. Youngs; University of Richmond: Mr. F. B. Key; Agricultural and Mechanical College of Texas: Mr. W. B. Coleman; University of Texas: Dr. J. P. LaSalle; United States Naval Academy: Mr. T. J. Benac, Dr. E. E. Betz; Wells College: Miss Mary D. Clement; State Teachers College, River Falls, Wisconsin: Mr. C. J. Kirchen; University of Wisconsin: Mr. H. N. Laden; College of Wooster: Mr. H. L. Meyer; Yale University: Dr. R. P. Dilworth.

Dr. Kazimierz Bartel, formerly professor of mathematics at the

University of Lwow and at the Polytechnic Institute, is reported to have been killed.

The death of Mr. John Brill, London, England, has been reported.

Professor Emeritus Alfred Pringsheim of the University of Munich died in Zurich on June 25, 1941 at the age of ninety-one years.

Professor Emeritus W. H. Butts of the University of Michigan died June 25, 1941.

Professor O. C. Edwards of the University of Minnesota died July 17, 1941.

Professor Emeritus J. W. Glover of the University of Michigan died July 15, 1941 at the age of seventy-two years. He had been a member of the Society since 1896.

The death of Mr. Pomeroy Ladue has been reported. He had been a member of the Society since 1894.

Professor C. E. Magnusson of the University of Washington died July 10, 1941. He had been a member of the Society since 1926.

Professor Emeritus Alice Winbigler of Monmouth College died May 27, 1941.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

454. R. H. Bruck and T. L. Wade: *The number of independent components of the tensor* ${}_{[\alpha]}T_{i_1 \dots i_p}$.

Let $T_{i_1 \dots i_p} = \sum_{[\alpha]} {}_{[\alpha]}T_{i_1 \dots i_p}$ represent the decomposition of an arbitrary tensor $T_{i_1 \dots i_p}$ into tensors of various symmetry types, a type corresponding to each partition $[\alpha]$ of the indices $i_1 \dots i_p$ (H. Weyl, *The Classical Groups*, pp. 96-136; T. L. Wade *Tensor algebra and Young's symmetry operators*, American Journal of Mathematics, vol. 63 (1941) pp. 645-657). J. A. Schouten (*Der Ricci-Kalkül*, chap. 7, §7) considers the problem of finding the number of independent scalar components of c_α of the tensor ${}_{[\alpha]}T_{i_1 \dots i_p}$ and obtains expressions for c_α in terms of the dimension n of the coordinate system in the cases $p=2, 3, 4$, but the difficulties of his method become great for $p \geq 5$. In this note it is shown that c_α is equal to r_α , the rank, recently defined by the authors, of the immanent tensor ${}_{[\alpha]}I_{j_1 \dots j_p}^{i_1 \dots i_p}$ (abstract 47-5-185). The general formulas for the r_α 's in terms of n may be obtained directly from the character table of the symmetric group on p letters whenever this is available, as for $p \leq 13$. Tables of $c_\alpha = r_\alpha$ are given for $p \leq 6$. (Received October 1, 1941.)

455. A. D. Campbell: *On the application of an algebra of sets to group theory.*

In this paper is assumed the existence of a set S of abstract entities (including 0 and 1) whose subsets are used as coordinates for groups, subgroups, elements of groups, sets of elements, and neighborhoods (in the study of topological groups). The relation $\alpha \subset \beta$ means that the element (or set or group or neighborhood) α is contained in the set (or group or neighborhood) β . By (α, β) is meant the join of α and β , by $\alpha \cdot \beta$ the meet of α and β . The unit element is called 1, the null-set (or null-group) is called 0. By $\alpha\beta$ is meant the set consisting of all products of elements of α by elements of β (in this order). Obvious meanings are given to $\alpha - \beta$ and α^{-1} . It is agreed that $[\alpha_1, \alpha_2, \dots, \alpha_n]$ shall mean the group generated by $\alpha_1, \alpha_2, \dots, \alpha_n$ and that $0\alpha = \alpha 0 = 0$, $1\alpha = \alpha 1 = \alpha$, $\alpha \cdot 0 = 0$, $\alpha - 0 = \alpha$. It is to be noted that $(\alpha, \beta)\gamma = (\alpha\gamma, \beta\gamma)$ and $(\alpha - \beta)\gamma = \alpha\gamma - \beta\gamma$ and $(\alpha \cdot \beta) \cdot \gamma = (\alpha\gamma) \cdot (\beta\gamma)$; also that for a group $\alpha\alpha = \alpha$, $\alpha\alpha^{-1} = \alpha$, $\alpha^{-1} = \alpha$. By this algebra old theorems are readily proved and new ones are derived. Thus it follows that if $\alpha = (\beta, \gamma)$ is a finite group of order a with β as a subgroup of order b and with $\beta \cdot \gamma = 0$, then the relation $\beta\alpha = (\beta\beta, \beta\gamma) = (\beta, \beta\gamma)$ shows clearly (since $\beta \cdot (\beta\gamma) = 0$) the well known result that $a = b + bc = b(1 + c)$. (Received September 15, 1941.)

456. Tomlinson Fort: *Generalizations of the Bernoulli polynomials and numbers and corresponding summation formulas.*

The operators d/dx and $\sum$ can be used to define the Bernoulli polynomials. These are generalized to linear operators by means of which very general classes of polynomials are defined, the coefficients of which serve to generalize the Bernoulli numbers. The polynomials determined are very general and include as special cases not only the classical Bernoulli polynomials, but the Bernoulli polynomials of the second kind, the Bernoulli and Euler polynomials of higher order as defined by Nörlund (*Differenzenrechnung*, p. 119) and the Bernoulli numbers as generalized by Vandiver (Proceedings of the National Academy of Sciences, vol. 23, p. 555). Attention is called to several new and interesting classes of polynomials and corresponding numbers. The polynomials of the paper are made the starting point for summation formulas analogous to Taylor's formula and to the Euler-Maclaurin formula of classical mathematics. A form for the remainder is obtained analogous to the Lagrange form for the remainder in Taylor's formula. (Received September 18, 1941.)

457. D. H. Lehmer: *Properties of the coefficients of the modular invariant $J(\tau)$.*

The properties of the integer coefficients in the Fourier series development of Klein's absolute elliptic modular invariant $j(\tau) = 12^3 J(\tau) = e^{-2\pi i \tau} + 744 + 196884e^{2\pi i \tau} + \dots = \sum_{n=-1}^{\infty} c(n) \exp(2\pi i n \tau)$ are of two sorts: (a) congruence properties with respect to small moduli and (b) multiplicative properties. The congruence properties are similar to those of Ramanujan's numerical function $\tau(n)$ with which the c 's are intimately related. For example, $(k+1)c(k) \equiv 0 \pmod{24}$. Multiplicative properties are discovered which enable one to express $c(kn)$ simply in terms of previous c 's for $n < k+1$. For example, $c(10) = c(6) + c(1)c(4) + c(2)c(3)$. Similar properties hold for the coefficients of any positive integral power of $j(\tau)$. (Received September 30, 1941.)

458. Howard Levi: *A characterization of polynomial rings by means of order relations.*

In this paper rings are considered for whose elements certain order relations and algorithms have been assumed. The assumptions made were borrowed from polynomial rings, where they appear as fairly obvious facts. It is shown that these facts characterize polynomial rings; that any ring for which they are postulated must be a polynomial ring. In view of the fact that polynomial rings with different types of coefficient domains differ significantly, two lists of assumptions are presented. The first leads to a ring of polynomials whose coefficients constitute a domain of integrity which contains a unit element and in which the Hilbert basis theorem holds. The coefficient domain derived with the second list is a field. These two lists of assumptions differ only slightly, and minor modifications of them could be made which would lead to more general types of coefficients. These cases are not discussed, in the belief that a polynomial whose coefficients are of a very general type has few claims to attention which it does not share with any element of a general ring. (Received September 25, 1941.)

459. R. J. Levit: *Fields in terms of a single operation.*

Though it is customary to consider fields as based on two independent operations, they may equally well be regarded as systems of single composition. This was first

shown by N. Wiener when he exhibited a set of seven postulates for fields in terms of a single operation. In the present paper are presented two other definitions of fields each in terms of a single operation and requiring five and six postulates respectively. The former takes as primitive an operation expressible as $a(1-b)$, which has the advantage of being class-closing. The latter definition uses Wiener's operation interpreted as $1-a/b$, which, though not class-closing, makes it possible to dispense with all existence postulates outside of the closure conditions. Detailed proofs are supplied of the necessity, sufficiency, consistency, and independence of both sets. A number of other single operations that can be used to define fields are also discussed. It is shown that no field-defining operation (or set of operations) is uniquely characterized by the postulates it satisfies. Each such operation (or set) is susceptible of a variety of interpretations, and all such interpretations are found. (Received October 1, 1941.)

460. Ernst Snapper: *Co-maximal linear sets and products of linear sets.*

The linear subsets of an n -dimensional vector space V_n satisfy the Noether decomposition theory (this Bulletin, abstract 47-3-124). The following three definitions are introduced: The product of two linear sets L_1 and L_2 is $(\mathfrak{E}_1 L_2 + \mathfrak{E}_2 L_1)$, where $\mathfrak{E}_i$ is the essential ideal of L_i . Two linear sets are co-maximal if their sum is the whole vector space. Two linear sets are strongly co-maximal if the sum of their essential ideals is the whole scalar domain. Strongly co-maximal linear sets are co-maximal. The definitions become the familiar definitions of co-maximal ideals and products of ideals if $n=1$. The following theorems are proved: The intersection of a finite number of linear sets, strongly co-maximal in pairs, equals their product. Every linear set is the irredundant intersection of a finite number of linear sets, strongly co-maximal in pairs; the intersection components themselves are not irredundant intersections of strongly co-maximal pairs and are unique. The results are used to study the decomposition of the restclass group V_n/L into a direct sum of subgroups. (Received September 8, 1941.)

461. L. R. Wilcox: *Extensions of semi-modular lattices. II.*

This paper generalizes a theorem of the author concerning complemented semi-modular lattices L (abstract 47-5-208) to the case where L is of dimension greater than or equal to 4 (that is, there exists in L a chain $a_1 > a_2 > \dots > a_n$, where $n \geq 6$), and L possesses no two parallel hyperplanes. (Received August 7, 1941.)

462. R. E. O'Connor: *A theorem of Diophantine approximation and an application to the values of a linear form.*

Assume θ given, $\eta \geq 1$, x, y rational integers and all other numbers real. Let the proposition P_1 assert that $c=c(\theta)$ exists such that, for every $t > 1$ and every α , the non-homogeneous system, $0 < x < t^\eta$, $|x\theta - y - \alpha| < t^{-1}$, has a solution (x, y) . Let P_2 assert that a positive γ exists such that, for every $t > 1$, the homogeneous system, $0 < x < \gamma t^{1/\eta}$, $|x\theta - y| < t^{-1}$, is insoluble. Referring to the standard classification of irrationals (see Koksma, *Diophantische Approximationen*, p. 27), it is easily shown that P_1 is equivalent to the proposition that θ is irrational and of type lower than $1/\eta$. In the present paper it is established that P_1 and P_2 are equivalent propositions, a theorem proved in 1924 by A. Khintchine for the case $\eta=1$. Now let ω, ω' be positive numbers

with ratio θ and let $\mu_0=0, \mu_1, \mu_2, \dots$ be the sequence S_μ of values, in order of increasing magnitude, taken by the form $x\omega+y\omega'$ as x, y range through the non-negative integers. For every positive μ , define $\Delta(\mu) = \max_{\mu_{i+1} > \mu} (\mu_{i+1} - \mu_i)$. Employing the above theorems it is proved that $\Delta(\mu) = O(\mu^{-1/\eta})$ ($\mu \rightarrow \infty$), if and only if θ is irrational and of type lower than $I\eta \infty$. Further information concerning the sequence S_μ is also given and extensions of the last theorem to cases with θ an irrational of higher type. (Received August 21, 1941.)

463. R. E. O'Connor: *Representation of integers by power-products of two integers.*

With a, b integers, $1 < a < b$, let $A(n)$ be the number of representations of the positive integer n in the form $n = \sum a_i a^{i_1} b^{i_2}$, the coefficients being non-negative integers less than a . Then $A(n)$ is equal to $P(n) = P(b, n)$, the number of partitions of n into non-negative powers of b . By a modification of a method of Hardy and Ramanujan (Proceedings of the London Mathematical Society, (2), vol. 17 (1918), pp. 75-115) it is shown that $\log P(n) \sim (\log n)^2 / 2 \log b$, as $n \rightarrow \infty$. The same result is established by showing, with $f(x) = 1 + \sum_{n=1}^{\infty} P(n) x^n$ ($|x| < 1$), that $\log f(x) \sim (\log^2 (1-x)) / 2 \log b$ ($x \rightarrow 1-0$) and by using this asymptotic value of $\log f(x)$ in the following theorem of Tauberian type. Assume: (1) $\phi(t) = a_0 + a_1 e^{-t} + a_2 e^{-2t} + \dots$, $a_i \geq 0$ ($i=0, 1, 2, \dots$), converges for $t > 0$; (2) $\log \phi(t) = (1+o(1))K(\log(t^{-1}))^\alpha$ ($t \rightarrow 0$), $K > 0$, $\alpha > 0$; (3) $A_n = \sum_{i=0}^n a_i$ ($n=0, 1, 2, \dots$). Then $\log A_n \sim (1+o(1))K(\log n)^\alpha$ ($n \rightarrow \infty$). (Received August 21, 1941.)

464. R. E. O'Connor: *Representation of integers by power-products of two real numbers.*

Let α, β be real numbers greater than unity with $\theta = \log \alpha / \log \beta$ irrational. Define $M(n)$ as the greatest integer such that every integer M' , $0 < M' \leq M$, has a representation of the form $M' = [\alpha^{r_1} \beta^{s_1}] + [\alpha^{r_2} \beta^{s_2}] + \dots + [\alpha^{r_v} \beta^{s_v}]$ where the r_i, s_i are non-negative integers and $v \leq n$. Very simple considerations show that to any $\epsilon > 0$ there corresponds an $n_0 = n_0(\epsilon, \alpha, \beta)$ such that $\log M(n) < (2+\epsilon)n \log n$, for $n > n_0$. Lower bounds for $M(n)$ are obtained in two cases as follows. Assume θ is of type lower than $I\eta \infty$; then for every positive k there is an $n_0 = n_0(k, \alpha, \beta)$ such that $\log M(n) > \eta^{-1} n \log n + kn$ ($n > n_0$). Or, assume α, β algebraic; then for every positive ϵ there is an $n_0 = n_0(\epsilon, \alpha, \beta)$ such that $\log M(n) > n(\log n)^{1/\epsilon - \epsilon}$ ($n > n_0$). In establishing the first of these lower bounds use is made of the theorems of abstract 47-11-462; for the second is required a theorem of Gelfond (Bulletin de l'Académie des Sciences de l'URSS, Série Mathématique, 1939, pp. 509-518). (Received August 21, 1941.)

465. Gordon Pall: *An elementary alternative to Dirichlet's theorem.*

It is shown how the application of Dirichlet's theorem on primes in an arithmetical progression can be replaced in many problems (especially concerning quadratic forms) by the application of Gauss's theorem on the existence of binary quadratic forms with assigned values for their generic invariants. For example, Meyer's theorem on indefinite quinary forms representing zero and the theorem giving conditions for quaternary zero forms can be proved by entirely elementary methods. (Received August 7, 1941.)

466. H. S. Vandiver: *An arithmetical theory of Bernoulli numbers.*

In this paper a method is described which enables one to find many new types of

arithmetical relations concerning the Bernoulli and allied numbers. It depends mainly on the following idea: Let a and b be rational, with $a \equiv b \pmod{p}$, where p is any prime integer. If a and b are independent of p it then follows, since there is an infinity of primes, that $a = b$. This is applied in connection with what is perhaps the simplest formula in which a single Bernoulli number appears: $S_n^1 = pb_n \pmod{p^2}$, where $S_n(p) = 1^n + 2^n + \cdots + (p-1)^n$. The paper starts with simple identities involving $(x^p - 1)/(x - 1)$, obtains congruences modulo p or p^2 by differentiation and integration, and then makes substitutions for the variables which give congruences involving the Bernoulli numbers. Generalizations of the Staudt-Clausen theorem as well as analogues of the same are obtained. Proofs are given of nearly all the results which were stated without proof in two previous papers by the writer (Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 555-559; vol. 25 (1939), pp. 197-201). This article will appear in the Transactions of this Society. (Received August 12, 1941.)

ANALYSIS

467. Lipman Bers: *A convergence theorem for analytic functions of two variables.*

Let M be a domain of the four-dimensional z_1, z_2 -space ($z_k = x_k + iy_k, k = 1, 2$) bounded by the two analytic hypersurfaces $E[z_1 \in B(z_2) + c(z_2), z_2 = e^{i\theta}, 0 \leq \theta < 2\pi]$ and $E[z_1 \in c(z_2), |z_2| < 1]$ where $B(z_2)$ is a star domain bounded by the curve $c(z_2) = E(z_1 = h(z_2, \lambda), 0 \leq \lambda < 2\pi]$ and $h(z_2, \lambda)$ is an analytic function of $z_2, |z_2| \leq 1$, for every fixed value of λ , and is subject to certain additional conditions. Consider a sequence of analytic functions $\{f_n(z_1, z_2)\}$ defined in M and satisfying the condition $\int_0^{2\pi} \int_0^{2\pi} |f(r_1 h(r_2 e^{i\theta}, \lambda), r_2 e^{i\theta})| r_1 dr_1 d\lambda < K, 0 < r_k < 1, k = 1, 2, n = 1, 2, \dots$, where p is a fixed number greater than 1. It is known that $f_n(z_1, z_2)$ possesses finite sectorial limits $(F_n(\theta, \lambda))$ almost everywhere on the intersection $F = E[z_1 = h(e^{i\theta}, \lambda), z_2 = e^{i\theta}, 0 \leq \theta < 2\pi, 0 \leq \lambda \leq 2\pi]$ of the two boundary hypersurfaces of M (see Bergman and Marcinkiewicz, *Fundamenta Mathematicae*, vol. 33 (1939), pp. 75-94, and a paper by the author to appear in the *American Journal of Mathematics*). If the sequence $\{F_n(\theta, \lambda)\}$ converges in a set of positive two-dimensional measure, the sequence $\{f_n(z_1, z_2)\}$ converges uniformly in every closed subdomain of M . This theorem can be generalized, in a modified form, for more general types of domains. (Received September 27, 1941.)

468. A. B. Brown: *Independent parameters for sets of functions.*

The results previously announced for the case of one function of n variables and m parameters (under the title *On the number of independent parameters*, abstract 46-11-485) are now extended to the case of a set of r functions of n variables and m parameters. The methods and results are similar to those for the case of one function. An additional theorem is given, on invariance of the number of parameters in a "complete" set under certain transformations of parameters. Only one paper is offered for publication, covering the more general case announced here, but under the original title. (Received September 11, 1941.)

469. E. R. Lorch: *The theory of analytic functions in normed abelian vector rings.* Preliminary report.

A complex vector space $\mathcal{R}$ for which the commutative multiplication of elements is defined is here called a normed abelian vector ring if the norm satisfies

$|A \cdot B| \leq |A| \cdot |B|$. $\mathcal{R}$ is assumed to possess a unit I for which $|I| = 1$. For those rings in which the singular elements (those without inverse) are precisely the frontier of the group of nonsingular elements, the classic theory of analytic functions is developed. Analytic functions are defined to be those mapping domains in $\mathcal{R}$ upon $\mathcal{R}$ for which the derivative exists. The theory elaborated for these functions includes the Cauchy theorem and formula and the power series representation. In addition, various known results on normed rings are derived anew by the author's methods. (Received September 3, 1941.)

470. A. N. Lowan and William Horenstein: *On the function*
 $H(m, a, x) = \exp(-ix)F(m+1-ia, 2m+2, 2ix)$.

The substitutions $\alpha = m+1-ia$, $\gamma = 2m+2$ and $u = 2ix$, in a known integral representation of the confluent hypergeometric function $F(\alpha, \gamma, x)$ led to the following interesting results: (1) A recurrence formula between the values of $H(m, a, x)$ for three consecutive integral values of the parameter m . (2) A recurrence formula between the values of $H(m, a, x)$ and $(\partial/\partial x)H(m, a, x)$ for two consecutive integral values of parameter m . (3) A recurrence formula between $H(m, a, x)$ and $(\partial/\partial x)H(m, a, x)$ for the integral value of the parameter m , and the value of $H(m+1, a, x)$. (4) A recurrence formula between the value of $H(m, a, x)$ for an integral value of the parameter m and the values of $H(m+1, a, x)$ and $(\partial/\partial x)H(m+1, a, x)$. The above results were obtained in the course of some exploratory study of the function $H(m, a, x)$ now being carried out by the Mathematical Tables Project conducted by the Work Projects Administration for the City of New York under the sponsorship of the National Bureau of Standards. (Received August 5, 1941.)

471. Brockway McMillan: *Absolutely monotone functions*. Preliminary report.

Let $\mathcal{A}$ be an additive complemented family of subsets of some space S . Let $\mathcal{A}^N$ be the additive complemented family determined by the Cartesian product sets $a_1 \otimes \cdots \otimes a_N$. Let $\mathcal{P}$ be the class of finite partitions π of S by pairwise disjoint members of $\mathcal{A}$, $\mathcal{P}$ being ordered by refinement. If $F(x)$ is defined for $x \in \mathcal{A}$, let $\Delta^0 F(x) = F(x)$ and $\Delta^N F(x; a_1 \otimes \cdots \otimes a_N) = \Delta^{N-1} F(x + a_N; a_1 \otimes \cdots \otimes a_{N-1}) - \Delta^{N-1} F(x; a_1 \otimes \cdots \otimes a_{N-1})$. If these differences are non-negative for every N and every collection of pairwise disjoint $a_i \in \mathcal{A}$ and if $u \in \mathcal{A}^N$, the limit on $\mathcal{P}$ $F^N(u) = \lim \{ \sum \Delta^N F(0; a_1 \otimes \cdots \otimes a_N \cdot u) | \pi \in \mathcal{P} \}$ exists for every N , the summation being taken over all combinations $a_1 \otimes \cdots \otimes a_N$ of distinct elements of π . If $G(x) = F^N(x^N)$, then $\Delta^{N+1} G(x; a_1 \otimes \cdots \otimes a_{N+1}) = 0$ identically in x and pairwise disjoint a_i . Furthermore, (i) $F(x) \geq F(0) + \sum (1/n!) F^N(x^N)$. If $\Delta^{K+1} F = 0$ identically, (i) is an equality and the right member terminates with the term $F^K(x^K)$. If (i) is an equality for some $x = x_0$, then equality holds for all $x \leq x_0$. There exist families $\mathcal{A}$ and functions F for which (i) is not identically an equality. (Received August 14, 1941.)

472. G. W. Mackey: *On the lattice of closed linear subspaces of a normed linear space*.

Let X_1 and X_2 be normed linear spaces. Let L_1 and L_2 be the lattices of closed linear subspaces of X_1 and X_2 , respectively. If L_1 and L_2 are isomorphic as lattices then, X_1 and X_2 are isomorphic as normed linear spaces, i.e., there exists a 1-1 linear transformation of all of X_1 into all of X_2 which is a homeomorphism. Eidelheit

(Studia Mathematica vol. 9 (1940) pp. 97-105) has proved an analogous theorem in which the ring of continuous linear transformations of the space into itself takes the place of the lattice of closed linear subspaces. He assumes in addition that X_1 and X_2 are complete. A strengthened form of Eidelheit's theorem in which it is not necessary to assume the completeness of X_1 and X_2 follows readily from the two lemmas on which the lattice theorem is based. (Received July 21, 1941.)

473. J. F. Paydon: *Convergence regions and value regions for continued fractions*. Preliminary report.

The author shows that if U is the circular region $|z| \leq r$, $r \leq 1/4$, and V is the circular region $|z - \alpha| \leq \beta$ where $\alpha = [1 + 2r - (1 - 4r)^{1/2}]/(4r + 2r^2)$, $\beta = [1 - r - (1 + r)(1 - 4r)^{1/2}]/(4r + 2r^2)$, and if $a_1, a_2, \dots$ lie in U , then the continued fraction $f = 1/1 + a_1/1 + a_2/1 + \dots$, which is necessarily convergent, has its value in V . Moreover if v is on the boundary of V , then there is one and only one continued fraction with elements in U whose value is v . This uniqueness property is also established for the parabolic element region and companion value region of Scott and Wall (abstract 46-11-503). The following convergence theorem is also proved: If the elements of f lie in the region $R(z) \geq 0$, $I(z) \geq 0$, $|z| - R(z) \leq 1$, then f converges if and only if (a) some a_n is 0, or (b) $a_n \neq 0$, ($n = 2, 3, 4, \dots$), and $\sum |b_n|$ diverges where, $b_1 = 1$, $a_n = 1/b_{n-1}b_n$, ($n = 2, 3, 4, \dots$). (Received August 7, 1941.)

474. Maxwell Reade and E. F. Beckenbach: *Square averages and a class of harmonic polynomials*.

Harmonic functions are characterized by the property that they are identical with their own circular averages, both areal and peripheral. The authors delineate the classes of functions which are identical with their own: (a) square averages, (b) rectangular averages. The first class is a class of harmonic polynomials, while the latter class is a more general class of polynomials. A typical result is the following: If the function $f(x, y)$ is superficially summable over a finite simply connected domain D , then a necessary and sufficient condition that $f(x, y) = (1/4h^2) \int_{-h}^h \int_{-h}^h f(x + \xi, y + \eta) d\xi d\eta$ hold for each point (x, y) in D , for all h sufficiently small, is that $f(x, y)$ be a harmonic polynomial of the form $f(x, y) = A_0 + A_1x + A_2y + A_3xy + A_4(x^2 - y^2) + A_5(x^3 - 3xy^2) + A_6(3x^2y - y^3) + A_7(x^2y - y^2x)$, where the A_i are constants. (Received August 27, 1941.)

475. J. F. Ritt: *Complete difference ideals*.

This paper appeared in full in the American Journal of Mathematics for October, 1941. (Received September 19, 1941.)

476. Rubin Schatten: *On the direct product of Banach spaces*.

Two Banach spaces E_1 and E_2 may be combined in two ways: the well known "sum" $E_1 \oplus E_2$, and $E_1 \otimes E_2$, where $\otimes$ refers to the distributive operation. The present paper deals with the second construction in its algebraic aspects, presents a theory of possible crossnorms, and considers the conjugate spaces and their crossnorms. The existence of a greatest crossnorm and a least crossnorm whose conjugate is also a crossnorm has been proved, and that they are conjugate to each other. A self-conjugate crossnorm has been constructed which is a generalization of the norm for Hilbert spaces given by F. J. Murray and John von Neumann (Annals of Mathematics, (2), vol. 37 (1936), p. 127). (Received September 26, 1941.)

477. H. S. Wall: *The behavior of certain Stieltjes continued fractions near the singular line.*

The author shows that the function $f(z)$ represented by a continued fraction of the form $1/1+g_1z/1+(1-g_1)g_2z/1+(1-g_2)g_3z/1+\dots$ in which $0 < g_n < 1$, $n=1, 2, 3, \dots$, and the series $\sum |1-2g_n|$ converges, approaches a finite limit $\alpha(s)$ as $z \rightarrow -s$, $s > 1$, along any path in the upper half-plane, and the limit $\bar{\alpha}(s)$, the complex conjugate of $\alpha(s)$, as $z \rightarrow -s$ from the lower half-plane. The function $\alpha(s)$ is continuous and not real, $s > 1$. (Received August 18, 1941.)

APPLIED MATHEMATICS

478. Arthur Korn: *On the problem of pulsating spheres in a liquid.*

The problem of the pulsating spheres treated for the first time almost 80 years ago by the Norwegian mathematician C. A. Bjerknes and chosen by the author as the point of departure for a mechanical theory of gravitation and the electromagnetic field, must be considered as a fundamental problem of mathematical physics. Interactions inversely proportional to the square of the distance can solely be produced by such spheres changing their volumes periodically with approximately the same frequency in an incompressible or approximately incompressible fluid. The sign of the forces of interaction (attraction in the case of equal phases, repulsion in the case of opposite phases) is a certain difficulty for the hypothesis that electrical particles are pulsating spheres. The sign can be changed, if one imposes on the spheres the condition that they maintain the amplitudes of their pulsations at any displacement of the spheres, a condition which is not imposed in Bjerknes' problem. The calculations are a little complicated, if one has to find out the pressure at the surfaces of the spheres and if one derives from these pressures the forces in question. Here is given a rather simple way of deriving the forces from the energy of the liquid in the case of displacements of the spheres, and one can easily understand the change of the sign, if one imposes on the spheres the condition that they maintain the amplitudes of their pulsations at any displacement. (Received September 22, 1941.)

GEOMETRY

479. George Comenetz: *Isotropic curves on a surface.*

Let S be a regular analytic surface in complex three-dimensional euclidean space, and O a point of S at which the tangent plane T is isotropic. Such a point is singular for the differential equation $ds^2=0$ of the isotropic net of curves on S . The problem is to describe the regular analytic isotropic curves on S that pass through O . Let L be the isotropic straight line in T through O , and p its order of contact with S . Draw an arbitrary curve F on S having contact of order p with L at O . Draw the normal to S at a point V of F , and let q be the "order of coincidence" of the normal with its limit L , as V approaches O . Then if $q > p/2$ the number of isotropic curves is 0 or 2, if $q = p/2$ it is 1 or 2 or $1 + \infty^1$, and if $q < p/2$ it is 2. The conditions under which the different possibilities are realized, and the orders of contact of the curves with L and one another, are stated. (Received September 15, 1941.)

480. J. J. DeCicco: *Bi-isothermal systems of curves.*

A generalization of the concept of isothermal family of curves in four-dimensional space S_4 is obtained in this paper. Kasner has termed the correspondences of S_4 , de-

finied by pairs of functions of two complex variables, the pseudo-conformal group G . This is characterized by the preservation of Kasner's pseudo-angle between a curve and a hypersurface at their common point. Any system of ∞^3 curves which is pseudo-conformally equivalent to a parallel set of lines is said to be bi-isothermal. Any such system consists of ∞^3 isothermal families, each family living on a conformal surface. A characteristic property of bi-isothermal systems is that the pseudo-angle between this system and any parallel pencil of hyperplanes is a biharmonic function. Finally G is characterized within the group of point transformations by the preservation of the totality of all bi-isothermal systems. (Received September 29, 1941.)

481. Walter Prenowitz: *Descriptive geometries as multigroups.*

Let G be the set of points of a descriptive space of arbitrary dimension. In G , define $a+b$ as the set of points between a and b if $a \neq b$, and $a+a$ as a . Then G becomes an abelian multigroup of special type. Convex sets and linear manifolds appear respectively as semigroups (subsets closed under $+$) and subgroups of G . Half-spaces (rays, half-planes, and so on) are cosets, when the latter are properly defined, and some of the elementary properties of half-spaces follow from a coset decomposition theorem. Elementary properties of these three types of figures are derived algebraically and many familiar group theoretic concepts as factor group, homomorphism, congruence relation are used in the development. (Received September 29, 1941.)

482. C. V. Robinson: *Spherical theorems of Helly type and congruence indices of spherical caps.*

The theorems mentioned in an earlier abstract (47-1-67) have been extended to the n -sphere. The principal theorem of Helly type reads: If each $n+k+2$ members of a family of convex subsets of the n -sphere intersect and if one member contains no k -dimensional great hypersphere then there is a point common to all the sets of the family. A study is also made of the congruence indices of spherical caps of various radii with respect to the class of semi-metric spaces. For example, it is found that a cap of spherical radius $\rho < \pi r/2$ of the 2-sphere of radius r will contain isometrically any semi-metric space of more than 6 points if every quadruple of the space is isometrically imbeddable in the cap, that is, the cap has the congruence indices $[4, 2]$. (Received September 23, 1941.)

STATISTICS AND PROBABILITY

483. J. F. Daly: *A problem in estimation.*

Consider a normal population in which each individual is characterized by the variates $y_1, \dots, y_p, y_{p+1}, y_{p+2}$. Suppose that the latter two are not directly observable, but that for given values of y_{p+1}, y_{p+2} the first set of y 's is independently distributed about the "regression line" $y_k = y_{p+1} + k y_{p+2}$ ($k = 1, \dots, p$) with a common variance σ^2 . For each individual, one can thus determine values $\hat{y}_{p+1}, \hat{y}_{p+2}$ from the observed $y_1, \dots, y_p$, using the method of least squares. Assuming a similar relation between the expected values of $y_1, \dots, y_{p+2}$ in the original population, these estimates $\hat{y}_{p+1}, \hat{y}_{p+2}$ are of course unbiased. However, if we calculate these $\hat{y}$'s for each individual of a sample of N and substitute them in the Pearson product-moment correlation formula the estimate of the correlation between y_{p+1} and y_{p+2} thus obtained is somewhat biased. The bias depends on the number of observable y 's and on the size of the variances and covariances of y_{p+1}, y_{p+2} relative to σ^2 . (Received September 2, 1941.)

484. Hilda P. Geiringer: *Some observations on analysis of variance theory.*

The distribution of the test functions used in the analysis of variance has been determined by R. A. Fisher for mutually independent chance variables subject to the same normal law. If it is not possible to determine the distributions of test functions for small samples and sufficiently general populations, the expectations and the variances of the test functions computed under appropriately general assumptions may be used. The expectations of the two quadratic forms known as "variance within" and "among" classes are equal even for non-normal but equal populations (Bernoulli series). Moreover it can be proved that the expectation of their quotient equals one. In case of Lexis and Poisson series, certain inequalities are investigated. These different criteria are completed by the computation of the variances. Also for some other symmetrical test functions general populations may be assumed. The Lexis as well as the Poisson series may then be characterized by equalities. Finally, omitting the restriction to independent chance variables, different kinds of mutual dependence are studied. These investigations which lead to new inequalities among the expectations seem to be related to Fisher's intraclass correlation and to supplement this idea. (Received August 11, 1941.)

485. E. J. Gumbel: *Simple tests for given statistical hypotheses.*

The probability integral transformation reduces the comparison of an observed distribution with a theoretical continuous distribution to the comparison of n observed points (n being the number of observations) with an equidistribution over the interval 0, 1. The well known shortcomings of the χ^2 test, the dependency on the classification and the necessity to combine the contents of the extreme cells can thus be avoided. The comparison of the observed and the theoretical means and measures of dispersion is very simple. The theoretical cumulative frequency of the m th point is the straight line $y = m/(n+1)$ where y , the mean position of the m th point, is plotted as the abscissa and $m (= 1, 2, \dots, n)$, the serial number, as the ordinate. The observed points will be scattered about this straight line. To judge the significance of these deviations, two control curves are constructed: To each point on the straight line correspond points $(y - \sigma, m)$ and $(y + \sigma, m)$, where σ the standard deviation of the m th point is given by $\sigma(n+2)^{1/2} = (y(1-y))^{1/2}$. (Received September 27, 1941.)

486. Abraham Wald: *Large sample distribution of the likelihood ratio.*

The large sample distribution of the likelihood ratio has been derived by S. S. Wilks (Annals of Mathematical Statistics, vol. 9 (1938)) in the case of a linear composite hypothesis and under the assumption that the hypothesis to be tested is true. Here a general composite hypothesis is considered and the distribution in question is derived also in case that the hypothesis to be tested is not true. Let $f(x_1, \dots, x_p, \theta_1, \dots, \theta_k)$ be the joint probability density function of the variates $x_1, \dots, x_p$ involving k unknown parameters $\theta_1, \dots, \theta_k$. Denote by H_0 the hypothesis that the true parameter point $\theta = (\theta_1, \dots, \theta_k)$ satisfies the equations $\xi_1(\theta) = \dots = \xi_r(\theta) = 0$ ($r \leq k$). Denote by λ_n the likelihood ratio statistic for testing H_0 on the basis of n independent observations on $x_1, \dots, x_p$. It is shown that under certain restrictions on $f(x_1, \dots, x_p, \theta)$, $\xi_1(\theta)$, $\dots$, $\xi_r(\theta)$ one obtains $\lim_{n \rightarrow \infty} \{P(-2 \log \lambda_n < t | \theta) - P[Q_n(\theta) < t | \theta]\} = 0$ uniformly in t and θ , where $Q_n(\theta)$ denotes a certain quadratic

form of r normally distributed variates, and $P(s < t | \theta)$ denotes the probability that s is less than t calculated under the assumption that θ is true. (Received August 11, 1941.)

TOPOLOGY

487. Samuel Eilenberg and Saunders MacLane: *Infinite cycles and homologies.*

An abelian group E is an extension of G by H if G is a subgroup of E and H is the corresponding factor group. With a suitable definition of equivalence and addition the extensions of G by H form a group. The authors consider the q th homology group of an abstract complex with coefficients in a group G obtained using infinite cycles and purely formal boundaries. A direct sum decomposition of this group is given, the first member of which is isomorphic with the group of all homomorphic mappings of the q th integral cohomology group (finite chains and cocycles) into G and the second with the group of all abelian group extensions of G by the $(q+1)$ th cohomology group. Various consequences of this result are obtained. (Received September 4, 1941.)

488. R. L. Moore: *On continua with dendratomic subsets.*

It is shown that in order that a compact continuum should have dendratomic subsets it is necessary and sufficient that it should not be a web. It follows that every compact continuum which is not a triod has such subsets. In particular every compact irreducible continuum between two points has them. (Received September 5, 1941.)

489. N. E. Steenrod: *Topological methods for the construction of tensor functions.*

The space M' of point tensors (fixed order and weight) over a differentiable manifold M of class r is shown to be a differentiable manifold of class $r-1$ forming a fibre bundle over M in the sense of Whitney. If M'' is a submanifold of M' which is still a fibre bundle over M with fibres F'' , methods are given for attempting to ascertain if a tensor function of class $r-1$ exists with values in M'' . An approximation theorem reduces the problem to finding a function which is merely continuous. If h is the smallest integer such that the homotopy group $\pi_h(F'') \neq 0$, a function of the required kind may be defined over the h -dimensional skeleton M^h of M . Any such determines a cocycle c^{h+1} in M with coefficients in the groups $\pi_h(F'')$. It is necessary to use here a homology theory based on local coefficient groups connected by local isomorphisms. In order that a tensor of the required kind exist over M^{h+1} , it is necessary and sufficient that $c^{h+1} \sim 0$. The existence and properties of the characteristic cohomology class are also proved for the more general type of fibre space introduced by Hurewicz and the author. (Received August 5, 1941.)

490. Henry Wallman: *Dimension for general spaces.*

For separable metric spaces there is a body of theorems of deep geometric appeal centering around the concept of dimension. In line with recent trends it seems natural to inquire whether one can extend the domain of application of dimension to spaces of more general character. The three common dimension functions are the Menger-Urysohn dimension d_1 , the Čech dimension-in-the-large d_2 and the Lebesgue covering dimension d_3 . It turns out that although there is not the slightest difficulty in applying these dimension-functions to spaces of the most general sort, one can hardly

proceed thereafter to build up a dimension theory for such general spaces with these for: (a) there exists a countable set for which d_1 is positive, (b) neither d_2 nor d_3 is monotonic ($A \subset B$ does not imply $d_2(A) \leq d_2(B)$ nor $d_3(A) \leq d_3(B)$) even for subsets of so regular a space as a bicomact Hausdorff space, (c) d_1, d_2, d_3 , do not coincide. (Upon the coincidence of d_1, d_2 , and d_3 for separable metric spaces rests a very great deal of the power of the dimension concept.) (Received September 3, 1941.)

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SOME APPLICATIONS OF THE RAYLEIGH-RITZ METHOD TO THE THEORY OF THE STRUCTURE OF MATTER¹

HUBERT M. JAMES

I have been asked to discuss for you some applications of the Rayleigh-Ritz method in physics. Rather than attempt to survey the whole of a discouragingly large field I have preferred to restrict my attention to some few topics from the theory of atomic and molecular structure. The application of the Rayleigh-Ritz method in this field has been carried out with some care and persistence, and there are enough results available to make possible a critical discussion of the potentialities and limitations of the method. This field also presents good examples of how one's mathematical procedure is influenced by one's physical notions concerning the system to be studied, and, in turn, what sort of physical conclusions may be drawn from results yielded by the method.

First of all I shall indicate why recourse to the Ritz² method is necessary if one is to obtain a satisfactorily accurate treatment of some of the principal problems in atomic and molecular theory. I shall indicate, though not in as much detail as I should like, the nature of the considerations by which one chooses the set of coordinate functions employed in this method, and show how rapid a convergence one can hope for in the approach to the solution. After presentation of some of the very satisfactory results one can obtain, by the application of enough hard work, in not too unfavorable cases, I shall finally have to indicate the increasingly serious limitations of the method as one seeks to apply it to the treatment of more and more complicated physical systems.

The fundamental mathematical problem in the study of an atomic system by wave mechanics is usually the solution of the stationary state wave equation for that system. In the non-relativistic case this is a linear partial differential equation of the second order, which can be symbolized thus:

$$H\psi = E\psi.$$

Here H represents the Hamiltonian operator for the system obtained

¹ An address delivered before the meeting of the Society in Washington, D. C., on May 3, 1941, by invitation of the Program Committee.

² W. Ritz, *Journal für die reine und angewandte Mathematik*, vol. 135 (1909), p. 1.

from the Hamiltonian function H of classical mechanics by the operator substitution

$$p_i \rightarrow \frac{h}{2\pi i} \frac{\partial}{\partial q_i}$$

where h is Planck's constant. For instance, for a hydrogen atom with fixed nucleus H is the sum of the kinetic energy of the one electron and the potential energy of the electron with respect to the nucleus:

$$H = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) - \frac{e^2}{r}.$$

The Hamiltonian operator correspondingly involves the Laplacian operator in a term representing the kinetic energy of the electron and a scalar multiplier representing the potential energy:

$$H = -\frac{h^2}{8\pi^2m} \nabla^2 - \frac{e^2}{r}.$$

The wave equation is then

$$-\frac{h^2}{8\pi^2m} \nabla^2 \psi - \frac{e^2}{r} \psi = E\psi.$$

In these equations E is a number, the energy of the system, and ψ , the corresponding solution, is the wave function which describes the state of the system when it has that energy. ψ is obviously a function of all the spatial coordinates needed to define a configuration of the system. In the case of a simple molecule like Cl_2 , for instance, ψ would depend on 108 variables. It is this multiplicity of variables which makes it so hard for physicists to interest mathematicians in some problems they would like very much to have solved.

Now not all the solutions of the wave equation are of interest, but only those which can be physically interpreted. This requirement of physical interpretability places certain auxiliary conditions on the functions ψ , often merely that they be quadratically integrable or that they remain bounded. Between the *physically interpretable* wave functions and the stationary states in which the system may be found there is a one-to-one correspondence. Thus in many cases physically satisfactory solutions of the wave equation exist only for certain discrete values of E , and in such cases one finds that the physical system also can possess only these discrete energies.

The physically satisfactory solutions ψ of such an equation and the

corresponding energies E form the raw material for any detailed quantitative treatment of the atomic system. For instance, a molecule can emit radiation when its energy changes from one of the discrete possible values to another. One can compute the average intensity of such radiation as soon as one knows the ψ functions for the initial and final stationary states, while the wavelength can be determined at once from the corresponding E 's. Thus by determining all the physically acceptable ψ 's and E 's which satisfy the wave equation for a molecule one could make a complete prediction of the wavelengths and intensities in the spectrum emitted by a gas consisting of these molecules, under any specified conditions. These wave functions are similarly fundamental in the study of collisions and other interactions between atoms or molecules, the absorption and scattering of radiation, and so on. The difficulty is to get reasonably accurate ψ functions on which to base further work.

In the simplest case of the hydrogen atom a complete solution of the problem is possible. The wave equation is separable in spherical coordinates, and one has to deal only with ordinary differential equations which offer no special difficulty. However, when one passes to the next more complicated problem, the helium atom with a heavy nucleus and two electrons, one meets most of the difficulties characteristic of this work.

If we treat the nucleus of the atom as fixed, the wave equation includes terms representing the kinetic energies of the electrons, the potential energies of the first and second electrons with respect to the nucleus, and their potential energy with respect to each other:

$$-\frac{h^2}{8\pi^2m}(\nabla_1^2 + \nabla_2^2)\psi(r_1, r_2) + \left(-\frac{2e^2}{r_1} - \frac{2e^2}{r_2} + \frac{e^2}{r_{12}}\right)\psi(r_1, r_2) \\ = E\psi(r_1, r_2).$$

It involves six independent variables or two vectors r_1, r_2 , defining the positions of the two electrons; one might, for instance, take as three of these variables the separations r_1, r_2, r_{12} of the three particles, on which the potential energy terms depend, and as the other three variables the angles necessary to define the orientation in space of the triangle formed by the three particles.

The fundamental difficulty of this equation arises from the fact that it is not completely separable. One can separate out the angle variables just mentioned, but there remains a partial differential equation in three variables which is not separable in any set of coordinates. In the very simplest case, in which ψ does not depend on the angle variables at all, the separated equation becomes

$$\begin{aligned}
& -\frac{\hbar^2}{8\pi^2m} \left(\frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2} + 2 \frac{\partial^2}{\partial r_{12}^2} + \frac{2}{r_1} \frac{\partial}{\partial r_1} + \frac{2}{r_2} \frac{\partial}{\partial r_2} + \frac{4}{r_{12}} \frac{\partial}{\partial r_{12}} \right. \\
& \quad \left. + \frac{r_1^2 - r_2^2 + r_{12}^2}{r_1 r_{12}} \frac{\partial^2}{\partial r_1 \partial r_{12}} + \frac{r_2^2 - r_1^2 + r_{12}^2}{r_2 r_{12}} \frac{\partial^2}{\partial r_2 \partial r_{12}} \right) \psi \\
& \quad + \left(-\frac{2e^2}{r_1} - \frac{2e^2}{r_2} + \frac{e^2}{r_{12}} \right) \psi = E\psi.
\end{aligned}$$

This equation is not only complicated, but unpleasantly singular, the potential energy becoming infinite as any one of the three variables approaches zero. Though it does not seem outside the range of possibility, no one has yet been able to set up a direct method for solving this equation or even for proving the existence of solutions. One must immediately resort to approximations.

One method by which physicists often deal with such equations is that of forcing a separation of variables by neglect of judiciously chosen small terms, the resulting ordinary differential equations then being treated by the customary methods. This and the closely related methods of Hartree³ and Fock⁴ actually serve to give useful approximate solutions of atomic wave equations involving several hundreds of variables. Peculiarly enough, the limitations of these methods are in practice apt to be more serious the lighter, the less complicated, the atom.

In the case of the state of helium with lowest energy, the ground state, computations by this method become quite simple. If one makes the very crude approximation of neglecting the mutual repulsion of the electrons, as represented by the term e^2/r_{12} , the approximate wave function becomes

$$\psi = Ce^{-2(r_1+r_2)},$$

the unit of length being the atomic unit $a_H = 5.28 \times 10^{-9}$ cm. The atomic energy computed with this function is -76.43 electron volts, against an observed value of -80.62 electron volts. Neglect of a group of terms,

$$\frac{e^2}{r_{12}} - \frac{5}{32} \left(\frac{e^2}{r_1} + \frac{e^2}{r_2} \right),$$

³ D. R. Hartree, *Proceedings of the Cambridge Philosophical Society*, vol. 24 (1928), p. 189; J. C. Slater, *Physical Review*, vol. 32 (1928), p. 339. Results published in many papers of Hartree, mostly in *Proceedings of the Royal Society of London*, series A.

⁴ V. Fock, *Zeitschrift für Physik*, vol. 61 (1930), p. 126.

which is on the average smaller than the single term e^2/r_{12} , leads to a better wave function, $\psi = Ce^{-(27/16)(r_1+r_2)}$, and a computed atomic energy in error by 1.53 electron volts. Further improvements are not easily obtained, and by pressing this method to its limit one can reduce the error in the computed energy only to 0.81 electron volts.

To obtain wave functions and energy values more accurate than these, as one very much wants to do, one must resort to the variational formulation of the problem. One then seeks to determine the wave functions, not as solutions of a differential equation, but as the properly continuous functions which extremalize the average energy

$$\bar{E} = \frac{\int \psi^* H \psi d\tau}{\int \psi^* \psi d\tau};$$

the corresponding extremal values of this quantity are then the stationary state energies of the system, which one wants to determine.

A variety of variational procedures have been applied in the theory of atomic structure, particularly in the study of the lowest energy states of the systems. The correct wave function for such a state makes the computed energy an absolute minimum. An approximate wave function will then presumably be better the lower—that is, the more nearly correct—the corresponding energy. It is accordingly a direct, and very common, procedure to set up approximate wave functions known to have a generally appropriate form, but involving adjustable parameters $\alpha, \beta, \dots$. Adjusting these parameters to minimize the average energy, one obtains the best approximate wave function of the assumed form, and possesses in the minimum value of $\bar{E}$ the most favorable upper limit on the ground state energy of the system.

For instance, one might assume a function of the form given by a forced separation of variables, but treat the constant in the exponential, which is the effective charge of the nucleus, as a variable parameter:

$$\psi(\alpha; r_1, r_2) = Ce^{-\alpha(r_1+r_2)}.$$

The best value of the parameter α is 27/16, the energy error being then 1.53 electron volts. Or one may set up the more elaborate and more flexible function,

$$\psi(\alpha, \beta; r_1, r_2) = C(e^{-\alpha r_1} e^{-\beta r_2} + e^{-\beta r_1} e^{-\alpha r_2}).$$

The best values of the parameters are then $\alpha = 2.15$, $\beta = 1.19$, and the error in the computed energy is only 0.76 electron volts, less than that given by any forced separation of variables.

Such schemes for the variation of parameters are often useful, but they have one serious limitation. In general an excessive amount of labor is required to determine the best values of the parameters, particularly if these occur in exponentials. As a result, the number of parameters which can be handled, and thus the flexibility of the trial functions, is seriously limited. This is one of the points at which the Rayleigh-Ritz method is superior to all others. Without excessive labor one can deal with Ritz functions containing as many as twenty parameters, whereas with other schemes one is in practice limited to two or three. The result is that the Ritz method is quite without a rival in the accurate computation of wave functions and atomic energies—as distinguished from roughly approximate computations.

The critical step in the application of the Ritz method is the choice of the set of coordinate functions in terms of which the wave functions are to be expressed as sums. Here one must balance against the need for reasonable ease in the computations the requirement of a general suitability in the character of the functions chosen, for even the considerable flexibility of the Ritz function in practice is not sufficient to compensate an inappropriate choice of the form of the component terms. One must bring to bear on this problem all the physical considerations which may be available. In the case of the ground state of helium the relatively simple arguments would run as follows:

First of all, since the electrons are identical the Hamiltonian operator is symmetrical to interchange of their coordinates. From this it follows easily that every He wave function should be either symmetric or antisymmetric to interchange of the coordinates of the electrons. States of the atom described by symmetric functions are those which the spectroscopist would class as singlet states, while the others are triplet states. The ground state of helium is known to be a singlet state; hence in attempting to determine its wave function one need consider only functions *symmetric to interchange of the electrons*. Similar general arguments,⁵ combined with the spectroscopic observation that the ground state of helium is an *S* state, show that this wave function does not depend at all on the angle variables, but *depends only on the distances between the particles*.

There are other arguments of a less precise character. For instance, if either electron of the atom is very far away from the nucleus the chance is overwhelming that the second electron will be, relatively, very near to it. The electric field at the first electron would then be nearly the same as if the second electron and the nucleus were super-

⁵ G. Breit, *Physical Review*, vol. 35 (1930), p. 569.

imposed—that is, almost Coulomb in character. Now we know that if the field were exactly Coulomb the wave function would decrease exponentially at large distances. Hence we can conclude that the situation will be similar in helium; the wave function should *decrease approximately exponentially* as the distance of either electron from the nucleus becomes very great.

This fact would be indicated by the approximate wave functions given by any of the methods previously discussed. But there is another important feature of the correct function which none of those approximate functions possesses; not one of them depends on the distance between the electrons. This is physically quite unreasonable, for it implies that the probability of any configuration of the system depends only on the electron-to-nucleus distances, and is just the same whether the electrons are very close together, on the same side of the nucleus, or far apart, on opposite sides of it. Obviously the mutual repulsion of the electrons will tend to keep them away from each other; correspondingly the wave function must tend to *decrease in magnitude with decreasing separation of the electrons*. It is the inability of most approximate wave functions to take account of this feature in the behavior of the system that is responsible for their largest errors.

How then shall one choose the terms from which the Ritz function is to be constructed? Formally, the simplest choice would be to make the Ritz function a power series, with the typical term,

$$r_1^m r_2^n r_{12}^p.$$

This is impossible, however, for no finite sum of such terms would show the desired exponential decrease for very large r_1 and r_2 ; the very integrals in terms of which the variational procedure is stated would not converge. But one can obtain the desired asymptotic character of the function, and assure the existence of all relevant integrals, by introducing an exponential factor into each term:

$$e^{-\delta(r_1+r_2)} r_1^m r_2^n r_{12}^p.$$

Introduction of an exponential factor in r_{12} might seem appropriate, but it would be quite unnecessary, and would only make the computations more tedious. A sum of such terms, in addition to the linear parameters characteristic of the Ritz method, would contain an exponential parameter for which it would be much more difficult to determine the very best value. Fortunately in practice one need worry little about the exact value assigned to such parameters; they may

be fixed once for all by an exploratory computation with a very few terms, or assigned a generally appropriate constant value on the basis of physical considerations of effective nuclear charges.

A finite sum of such terms can have all the properties one desires in respect to flexibility, asymptotic behavior and simplicity, but may lack the proper symmetry to interchange of the coordinates of the electrons. This symmetry one can obtain in various ways. For instance, one may construct the terms from powers of $r_1 + r_2$ and $r_1 - r_2$:

$$e^{-\delta(r_1+r_2)}(r_1 + r_2)^m(r_1 - r_2)^n r_{12}^p,$$

and assure the desired symmetry by including terms with all powers of $(r_1 + r_2)$, but only even powers of $r_1 - r_2$. These were just the coordinate functions used by Hylleraas in his pioneer work on the theory of the helium atom.⁶

In his first work Hylleraas obtained a quite simple six-term function,

$$\psi = e^{-1.818(r_1+r_2)} \{c_0 + c_1 r_{12} + c_2 r_{12}^3 + c_3(r_1 + r_2) + c_4(r_1 + r_2)^2 + c_5(r_1 - r_2)^2\}.$$

In the atomic units $R_H h$ the energy computed with this function is -5.80633 , against the observed value of $-5.80736 \pm .00012$; the error of .0139 electron volts is about one-sixtieth that of the best previous computation. In later work⁷ with an eight-term function the computed energy was depressed to -5.80748 . It might seem embarrassing to the theory that the computed upper limit on the energy should come below the observed energy. However, the wave equation treated by Hylleraas involved certain approximations, neglecting relativistic effects and in part, the motion of the nucleus. Adding a correction of $+0.00010$ for the effect of nuclear motion and one of $+0.00018$ for relativistic effects⁸ one obtains a theoretical value of -5.80720 for the ground state energy of helium. This agrees with the observed value to one part in 30,000, practically within the accuracy of observation, and, as was to be expected, lies slightly too high.

This very satisfactory treatment of the three-body problem in wave mechanics is of historical interest in that it gave the first quantitative check on the present theory of atomic structure as applied to systems more complicated than the hydrogen atom. Many theories have been devised to account for the very simple properties of hydro-

⁶ E. A. Hylleraas, *Zeitschrift für Physik*, vol. 54 (1929), p. 347.

⁷ E. A. Hylleraas, *Zeitschrift für Physik*, vol. 65 (1930), p. 209.

⁸ H. Bethe, *Handbuch der Physik*, XXIV/1. See pages 371, 383.

gen, but no other one of them has been able to stand the test of extension to the treatment of helium. The confidence of the physicist in the essential correctness of the present atomic theory is of course based on a wide variety of comparisons with experiment. A number of other equally impressive examples of this particular quantitative type have been obtained through application of the Ritz method, and will be presented later.

By making use of the observed energies of the higher stationary states of helium one can fix an upper limit on the error of these wave functions.⁹ One can show, for instance, that the root-mean-square error in the six-term function does not exceed 3 per cent; a reasonable estimate is 1 per cent. For the more complicated function the error is of the order of 1/5 per cent. Such accuracy is more than adequate for the purposes of the physicist.

Similar results can be obtained in studies of the lower excited states of helium. These states differ from the ground state in one important physical respect. In the ground state the two electrons are said to be equivalent; the most probable configurations of the atom are those in which the two electrons are at about the same distance from the nucleus. In the lower excited states of the atom just one of the electrons is excited; the most probable configurations are those in which one electron is much farther from the nucleus than the other. The effective nuclear charges are then different for the two electrons, that of the outer electron being reduced by its repulsion by the inner electron. This one should take into account in constructing the Ritz function, replacing

$$e^{-\delta(r_1+r_2)} \frac{m}{r_1} \frac{n}{r_2} \frac{p}{r_{12}}$$

by

$$e^{-\delta_1 r_1} e^{-\delta_2 r_2} \frac{m}{r_1} \frac{n}{r_1} \frac{p}{r_{12}}.$$

The terms in the Ritz function are then to be constructed by making appropriate linear combinations of these functions, symmetric in the case of ^1S terms, and antisymmetric in the case of ^3S terms.

If the excited state to be considered is not the lowest of its symmetry one does not obtain the energy of the state as the lowest root of a secular equation, but as the second lowest root, third lowest root, and so on. Fortunately one can show that an approximate energy thus computed is always an upper limit for the energy of the corresponding state of the atom, approaching the correct value from above as the

⁹ H. M. James and A. S. Coolidge, *Physical Review*, vol. 51 (1937), p. 860.

flexibility and accuracy of the Ritz function are increased.^{10,11} Consequently the error in the computed energy provides an indication (which can be made quantitative) of the accuracy of the corresponding function. This is an outstanding advantage of the Ritz method over all other practical methods of treating excited states, in which an indifferent approximate wave function can by chance lead to the correct value of the energy.

Among the results of computations on excited states of helium are the following: The lowest excited state of helium is a ^3S state. The observed energy is -4.3505 in the atomic units $R_H h$ while Hylleraas and Undheim,¹⁰ using a six-term function, have computed an energy of -4.3504 , an error of about one part in 40,000. The second excited state is a ^1S state, with an observed energy of -4.2920 , while Coolidge and James¹² have used an eleven-term function to compute the value -4.2916 , in error by one part in 10,000. The greater complexity of the function needed in the case of the more highly excited state will illustrate a general feature of computations with the Ritz method: convergence on the correct function becomes slower the more highly excited is the state considered. This factor, together with the increasing usefulness of certain approximate methods, limits the interest in extending Ritz computations to very highly excited states.

Another simple and important problem is that of the hydrogen molecule, H_2 . This provides illustrations of the results obtainable with the Ritz method in rather different and more complicated circumstances, and also material for some general remarks about the method, which can here be based on more extensive studies than in the case of helium.

The problem of treating H_2 , a system containing two heavy nuclei and two light electrons, can be reduced to that of solving the wave equation for the motion of the electrons in the field of the nuclei, considered as fixed. The wave equation is then

$$\left\{ -\frac{\hbar^2}{8\pi^2m} (\nabla_1^2 + \nabla_2^2) + \left(-\frac{e^2}{r_1} - \frac{e^2}{s_1} - \frac{e^2}{r_2} - \frac{e^2}{s_2} + \frac{e^2}{r_{12}} + \frac{e^2}{R} \right) \right\} \psi(R; r_1, r_2) \\ = E(R) \psi(R; r_1, r_2).$$

This partial differential equation involves the six coordinates of the two electrons as independent variables, and the fixed separation of the nuclei, R , as a parameter. It contains terms corresponding to the

¹⁰ E. A. Hylleraas and B. Undheim, *Zeitschrift für Physik*, vol. 65 (1930), p. 759.

¹¹ J. K. L. MacDonald, *Physical Review*, vol. 43 (1933), p. 830.

¹² A. S. Coolidge and H. M. James, *Physical Review*, vol. 49 (1936), p. 676.

kinetic energies of the first and second electrons, the potential energy of the first electron at a distance r_1 from the first nucleus and s_1 from the second nucleus, the potential energy of the second electron in the field of the nuclei, the potential energy of repulsion of the two electrons, and the potential energy of repulsion of the two nuclei. Again the difficulty in solving the wave equation arises from the non-separability of the variables. In this case at most one variable can be separated out, and every exact solution depends on at least five variables—for instance, $r_1, r_2, s_1, s_2, r_{12}$.

The physical considerations on which one can base approximate solutions of this equation are interesting but so extensive and complicated that I cannot undertake to present them here. I can suggest only one of several possible lines of thought. If, for instance, one neglects the repulsion of the electrons the problem becomes completely separable. To this approximation, each electron moves as it would in the absence of the second electron—that is, as if it were the one electron of an ionized hydrogen molecule. Correspondingly the wave function becomes a product of two functions, each a solution of the wave equation for H_2^+ .

$$\psi(R; r_1, r_2) = \chi_1(R; r_1)\chi_2(R; r_2),$$

$$\left\{ -\frac{\hbar^2}{8\pi^2m} \nabla_1^2 - \frac{e^2}{r_1} - \frac{e^2}{s_1} \right\} \chi(R; r_1) = \epsilon(R)\chi(R; r_1).$$

Now this latter equation is separable in elliptic coordinates $\lambda_1 = (r_1 + s_1)/R$, $\mu_1 = (r_1 - s_1)/R$, $\phi_1 =$ azimuth about the internuclear axis, and the form of the solutions can be determined in detail. In the cases to be mentioned later the interesting solutions are independent of ϕ and can be well approximated by a sum of a few terms of the form $e^{-\delta\lambda} \lambda^m \mu^j$. This suggests that one might well form the terms of a Ritz function for hydrogen as products of such factors, one for each electron, multiplied by various powers of the interelectronic distance to take account of the tendency of the wave function to decrease as the electrons approach each other: $e^{-\delta(\lambda_1 + \lambda_2)} \lambda_1^m \lambda_2^n \mu_1^j \mu_2^k r_{12}^p$. The symmetry or antisymmetry required of the function according as one treats singlet or triplet states of the system can be assured with only slight complication of the form of the terms: $e^{-\delta(\lambda_1 + \lambda_2)} (\lambda_1^m \lambda_2^n \mu_1^j \mu_2^k \pm \lambda_1^n \lambda_2^m \mu_1^k \mu_2^j) r_{12}^p$. Coolidge and I¹³ have used just such terms in treating the ground state of H_2 . For application to excited states of the molecule only one slight modification is necessary. Since the electrons are no longer

¹³ H. M. James and A. S. Coolidge, *Journal of Chemical Physics*, vol. 1 (1933), p. 825; *ibid.*, vol. 3 (1935), p. 129.

equivalent, one must introduce two different exponential parameters for the two electrons, corresponding to the different effective nuclear charges.

In working with these functions we have made extensive studies of the way in which the computed energy converges on the correct energy as more and more terms are added to the Ritz function. The convergence behavior of course varies with the chosen form of the terms, but when a physically appropriate choice has been made the convergence follows a remarkably similar pattern, whether one is working with some state of the hydrogen molecule, or the helium or lithium atoms, or the lithium molecule Li_2 , or any other system with which we have had any experience.

In a favorable case, of which the ground state of H_2 is typical, the convergence is very simple and regular. A presentable approximate function and energy value can be obtained with relatively few terms—say five or six. Thereafter one finds that the small improvements obtained on adding further terms to the function are nearly additive. Thus a term which produces an improvement of .01 volt in the computed energy when it is added to a five-term function will give nearly the same improvement when added, separately or together with other terms, to a ten-term function. Deviations from strict additivity of improvements are almost invariably in the direction of a reduction in the importance of a term when it is added to a better function. Thus it is possible to put an upper limit on the error arising from neglect of any given term by trying it out together with a small number of others. This greatly reduces the labor of eliminating useless terms from consideration and getting the simplest possible Ritz function of a given accuracy.

Another important feature of the convergence is the rapid and regular falling off in importance of the terms in a sequence in which the value of any single exponent is increased. Thus in one case the term with exponents (0, 0, 1, 0, 0) was one of the most important terms, (0, 1, 1, 0, 0) gave an improvement of .008 electron volts, and (0, 2, 1, 0, 0) an improvement of .0015 electron volts. This rapid convergence makes it relatively easy to estimate closely the improvement in energy to be obtained on adding to the Ritz function all the terms of a given sequence. The near independence of the improvements due to the different sequences then makes possible a useful estimate of the convergence limit as all terms of the complete set are added to the Ritz function. For example, we have obtained the following results in treating the ground state of H_2 . A five-term function gave the energy of the molecule as -2.3329 atomic units, against the observed value

of -2.3493 —a result already far more accurate than has been obtained by any method other than the Ritz method. A thirteen-term function gave the energy as -2.3470 , correct to 1/10 per cent; the root-mean-square error in that function itself is of the order of 1 per cent. Finally, a careful convergence estimate led to the energy value $-2.3492 \pm .0008$. When this value was published it was considerably more reliable than any available experimental result, but a later very accurate determination by Beutler and Jünger¹⁴ gave the experimental value quoted above, in complete agreement with the apparent convergence limit of the Ritz method.

In treating some of the excited states of H_2 we have encountered a less simple convergence behavior.¹⁵ Sometimes it happens that two terms will cooperate; that is, they will together improve the computed energy by more than the sum of the improvements they would produce separately. We have not encountered any case in which three or more terms have formed a cooperating group. When such cooperation occurs it is usually easy to understand in physical terms. In practice in treating a given problem one quickly finds out whether such cooperation is likely to occur and can predict fairly accurately what pairs of terms will show it. Somewhat more care is required in discarding terms and making convergence estimates, but the difficulty is not serious.

Somewhat more annoying is the general tendency toward slower convergence in the case of the excited states. With the ground state five terms suffice to give a very useful function, while with an excited state ten terms may be necessary to obtain similar accuracy. This increasing slowness of convergence is most marked when one deals with the higher states of a given symmetry. Thus our best function for the ground state, a $^1\Sigma_g$ state, contains thirteen terms. To get a somewhat less accurate result for the first excited state of this character sixteen terms were needed, and we have never succeeded in getting a usable approximation for the second excited $^1\Sigma_g$ state, which, incidentally, would be of particular physical interest.

Despite these difficulties the Ritz method has yielded very useful results for several excited states of H_2 .¹⁵ The maximum accuracy for practical work was obtained with functions containing about sixteen terms, after many other terms were tried and discarded as negligible. These functions are accurate to the order of 2 per cent, the computed energies to about 0.1 per cent.

¹⁴ H. Beutler and H.-O. Jünger, *Zeitschrift für Physik*, vol. 101 (1936), p. 304.

¹⁵ H. M. James, A. S. Coolidge and R. D. Present, *Journal of Chemical Physics*, vol. 4 (1936), p. 187; A. S. Coolidge and H. M. James, *ibid.*, vol. 6 (1938), p. 730.

The physical significance of such applications of the Ritz method may be illustrated by the results of a computation in which use was made of these wave functions.¹⁶ When H_2 is bombarded with electrons having more than 12 electron volts of energy it emits a strong continuous spectrum in the ultraviolet. This arises when molecules excited to the stable $1\sigma 2s\sigma \ ^3\Sigma_g$ state pass over into the lower and unstable $1\sigma 2p\sigma \ ^3\Sigma_u$ state. If the energy of the bombarding electrons is carefully kept to a minimum or if enough helium is present in the discharge this spectrum takes on a particularly simple and reproducible character, for the vibrations of the molecule in the initial electronic state are suppressed. Using Ritz wave functions for both electronic states of the molecule we have made a complete theoretical prediction of the intensities in this spectrum. The results appear in Figure 1, together with the observational results of Smith, the best

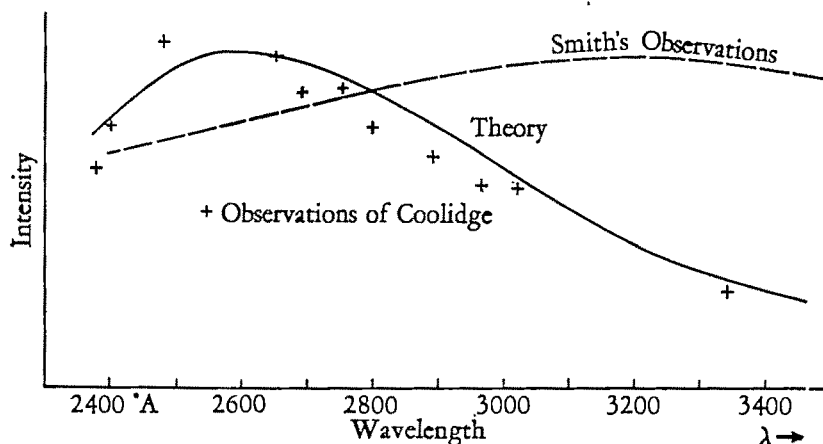


FIG. 1

available at the time. The agreement was not to be regarded as satisfactory. Since we had complete confidence in the theory we were forced to the conclusion that the experiments were in error, and that the intensity standards used by Smith were not satisfactory for work in this very difficult region of the spectrum. This has now been proved to be correct by the preliminary results of some very careful experimental work by Dr. Coolidge, also shown in Figure 1. On employing improved intensity standards, he finds that the gross disagreement of theory with experiment disappears. It is true that there remain discrepancies which are larger than one would like. This seems to be due

¹⁶ H. M. James and A. S. Coolidge, *Physical Review*, vol. 55 (1939), p. 184.

to difficulties with the intensity standard, which is still less reliable than the theoretical computations. It would not be at all unreasonable to work backward, from the theory and the observations of Coolidge, to an improved calibration of the standard source.

Now a somewhat less satisfactory aspect of this subject must be mentioned. While the helium atom and the hydrogen molecule are of rather special interest to physicists they are also almost the simplest systems of their respective types, much too simple to be entirely typical. Unfortunately one soon finds on attempting to extend the Ritz method to the treatment of other atomic systems that the difficulties are numerous and serious. To indicate their character I need only to pass to the case of lithium.

The lithium atom consists of a nucleus and three electrons. In the simplest possible case the wave functions depend on six essential and non-separable variables—say the six distances between pairs of particles. The first discouraging effect of this greater number of variables is the corresponding increase in the number of potentially important Ritz terms with all exponents small. In passing from the hydrogen molecule with five essential variables to the lithium atom with six there is actually an increase from eight to thirty-two of the Ritz terms in which no exponent exceeds 1. The number of sequences of terms to be investigated in making convergence estimates is correspondingly increased.

Each term is also more difficult to work with. The increased number of electrons makes the symmetry conditions more complicated,¹⁷ so that each Ritz term is, at the simplest, the sum of six parts, instead of the two parts needed with helium or hydrogen. There are thus more integrals to be evaluated in setting up the secular equation for the problem, and they are more difficult to evaluate because of the greater number and the awkward character of the variables of integration.

In the case of lithium it is possible to overcome these difficulties.¹⁸ The observed total energy of the lithium atom is -14.9578 atomic units. Using a 17-term function we have computed the energy -14.9566 , in error by one part in 10,000. Of more physical interest, perhaps, is the energy needed to ionize the atom. Here the experimental result is 5.364 e.v., while our convergence estimate leads to the theoretical value $5.363 \pm .007$ e.v. •

It does not seem useful, however, to apply the Ritz method in accurate treatments of atoms more complicated than lithium. The diffi-

¹⁷ H. M. James and A. S. Coolidge, *Physical Review*, vol. 55 (1939), p. 873.

¹⁸ H. M. James and A. S. Coolidge, *Physical Review*, vol. 49 (1936), p. 688.

culties increase enormously and dampen effectively any interest one may have in this method of studying the systems. In the case of molecules the difficulties are even greater; to give an accurate treatment of the molecule Li_2 is much more than twice as difficult as to give a corresponding treatment of the lithium atom. It is true that the Ritz method has been fruitfully applied to the treatment of LiH ,¹⁹ and of Li_2 ,²⁰ and especially of the ionized molecules¹⁹ LiH^+ and Li_2^+ ,²¹ but here again the limits to the extension of the method seem to have been reached.

In conclusion I should emphasize that this particular field of atomic physics has been discussed, not as a summary of all that has been accomplished in physics using the Rayleigh-Ritz method, but rather as an illustration of how the method is likely to work out in practice where it is applied. Applications have been made in widely varied fields—the theory of nuclear structure, acoustics, the electromagnetic theory of cavity resonators—to mention three. The field I have discussed is typical in many respects. One can expect in general that the Ritz method can yield extremely accurate results with some problems not tractable to other methods of computation, and that its application will become impracticable just in cases where very many independent variables must be considered, or boundary or other conditions are especially complicated. What is abnormal about this field is the relative completeness with which the Ritz method has been exploited in it. This is due just to the scarcity of atomic problems in which one can deal with a reasonably small number of independent variables. In the majority of fields one can expect to find an unending sequence of problems to which the method can be usefully applied.

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¹⁹ J. K. Knipp, *Journal of Chemical Physics*, vol. 4 (1936), p. 300.

²⁰ H. M. James, *Journal of Chemical Physics*, vol. 2 (1934), p. 794.

²¹ H. M. James, *Journal of Chemical Physics*, vol. 3 (1935), p. 9.

SOME APPLICATIONS OF CERTAIN POLYNOMIAL CLASSES¹

I. M. SHEFFER

The term *applications* will be construed broadly enough as to include *properties*. It is here proposed, then, to examine some properties and some applications of some classes of polynomial sets.

1. **Formal properties of Appell sets.** The power series approach to the theory of analytic functions presents us with a very simple polynomial set.² Thus, if $f(x)$ is analytic about $x=a$, we have the expansion

$$(1) \quad f(x) = \sum_{n=0}^{\infty} c_n \frac{(x-a)^n}{n!}, \quad c_n = f^{(n)}(a),$$

in terms of the set

$$\{(x-a)^n/n!\}.$$

This set has the important property of reproducing itself under the operation of differentiation, in accordance with the rule.

$$(2) \quad \frac{d}{dx} P_n(x) = P_{n-1}(x),$$

where

$$P_n = (x-a)^n/n!.$$

Now this set is not uniquely determined by (2). There are in fact infinitely many sets of polynomials $\{P_n\}$ that satisfy (2). These bear the name of *Appell sets*,³ after the man who in 1880 [1]⁴ first made a study of them. Appell sets will be the first class to be considered here.

There are many conditions that are equivalent to the defining relation (2) for Appell sets. Among the simplest are the following two:

¹ An address delivered before the meeting of the Society in Washington, D.C., May 3, 1941, by invitation of the Program Committee.

² By a set $\{P_n\}$ we understand an infinite sequence $P_0, P_1, \dots$, with P_n of degree n .

³ Strictly speaking, Appell's definition is

$$(2') \quad \frac{d}{dx} P_n(x) = nP_{n-1}(x),$$

but (2) is preferable for our purpose. If $\{P_n\}$ satisfies (2), then $\{nP_n\}$ satisfies (2'), and conversely. A bibliography on Appell polynomials is given in Davis [5].

⁴ Numbers in brackets refer to the bibliography placed at the end.

(i) A set $\{P_n\}$ is an Appell set if and only if constants $\{a_n\}$ exist ($a_0 \neq 0$) such that

$$(3) \quad P_n(x) = a_0 \frac{x^n}{n!} + a_1 \frac{x^{n-1}}{(n-1)!} + \cdots + a_n \frac{x^0}{0!}, \quad n = 0, 1, \dots$$

(This is the explicit representation of an Appell set.) (ii) In order that $\{P_n\}$ be an Appell set it is necessary and sufficient that a formal power series

$$(4) \quad A(t) = \sum_0^{\infty} a_n t^n$$

exist so that (formally)

$$(5) \quad A(t)e^{tx} = \sum_0^{\infty} P_n(x)t^n.$$

$A(t)$ will be called the *generating function* for $\{P_n\}$.

It is natural to inquire, regarding a set of polynomials, if it has properties analogous to those of the classical sets (Legendre, Hermite, and so on). For example, does it satisfy a linear differential equation of the form

$$L_0 y + L_1 y' + L_2 y'' = \lambda y$$

where L_i is a polynomial of degree not exceeding i , and λ is a parameter which takes on a value λ_n for the n th polynomial of the set? Or an equation of the same type but of higher order? Such equations contain too few constants to serve for *all* Appell sets, although we shall see that there is a subclass of such sets that satisfy equations of this type. If, however, we are willing to permit the order to become infinite:

$$(6) \quad \sum_{r=0}^{\infty} L_r(x) y^{(r)}(x) = \lambda y(x),$$

where L_r is of degree not exceeding r , then there is ample freedom not only for Appell sets but for all sets. That is, (6) is a *universal equation* for all polynomial sets in that given an *arbitrary* set $\{P_n\}$, it is possible to choose $\{L_r\}$, $\{\lambda_r\}$, and indeed in infinitely many ways, so that $\{P_n\}$ satisfies (6) ($y = P_n$ for $\lambda = \lambda_n$).

The advantages of a universal or, as we may say, *canonical* equation are evident. But this very generality may sometimes be a defect in studying particular sets, unless it is possible to characterize the coefficients $\{L_r\}$ corresponding to such sets. Fortunately a charac-

terization can be achieved in the case of Appell sets. The simplest (but not the only⁵) canonical equation of form (6) for Appell sets⁶ is [11]

$$(7) \quad (b_0 + x)P_n' + b_1P_n'' + b_2P_n''' + \cdots = nP_n, \quad n = 0, 1, \cdots$$

That is, $L_0 = 0$, $L_1 = b_0 + x$, $L_k = b_{k-1}$, $k \geq 2$; $\lambda_n = n$. Moreover, if $\{P_n\}$ has the generating function $A(t)$, then $\{b_n\}$ is defined by

$$(8) \quad \sum_0^\infty b_n t^n \equiv B(t) = A'(t)/A(t).$$

It follows from (7) that $\{P_n\}$ is an Appell set if and only if a sequence $\{b_n\}$ exists so that

$$(9) \quad nP_n(x) = (b_0 + x)P_{n-1} + b_1P_{n-2} + \cdots + b_{n-1}P_0, \quad n = 1, 2, \cdots$$

This is of interest because it gives the set $\{P_n\}$ by recurrence. It has been used by Webster [17] to give a simple proof that the Hermite polynomial set is the only Tchebycheff (orthogonal) set that is at the same time an Appell set. Here then is an Appell set that satisfies a second order equation of form (6). In general we have the theorem: *An Appell set $\{P_n\}$, with generating function $A(t)$, satisfies a finite order equation of type (6) if and only if $A(t)$ has the form*

$$(10) \quad A(t) = e^{Q(t)}, \quad Q(t) = \text{polynomial};$$

and the minimum order of all such equations for $\{P_n\}$ is the degree⁷ of $Q(t)$ [11]. As examples, the set $\{(x-a)^n/n!\}$ and the Hermite set have for $A(t)$ the respective functions e^{-at} , $e^{-t^2/4}$; and these sets satisfy equations of the first and second order respectively.

2. Relation of Appell sets to functional equations. In 1888 Pincherle made a study of the following linear difference equation with constant coefficients:

$$(11) \quad L[y(x)] \equiv \sum_{j=1}^k a_j y(x + \omega_j) = F(x).$$

His memoir, in a French translation, was reprinted in 1926 in *Acta Mathematica* [10]. In the course of his investigation he pointed out the value of Appell sets. Thus for (11), two Appell sets preëminently suggest themselves, namely $\{P_n\}$, $\{Q_n\}$, defined respectively by

⁵ There are infinitely many.

⁶ Conversely, if $\{P_n\}$ satisfies (7), then $\{c_n P_n\}$ will be an Appell set for proper choice of the constants $\{c_n\}$.

⁷ The one exception is when $Q(t) = \text{constant}$.

$$(12) \quad L[x^n/n!] = P_n(x), \quad L[Q_n(x)] = x^n/n!.$$

For as Pincherle observed, if $F(x)$ has the expansion

$$(13) \quad F(x) = \sum_0^{\infty} c_n P_n(x),$$

then a formal solution of (11) is given by

$$(14) \quad y(x) = \sum_0^{\infty} c_n x^n/n!;$$

and if $F(x)$ has the expansion (1) (for $a=0$), then

$$(15) \quad y(x) = \sum_0^{\infty} c_n Q_n(x)$$

formally satisfies (11). For the case that $F(x)$ is an entire function of finite exponential type,⁸ Pincherle showed that there is a solution of (11) of the same kind. He also found a solution when $F(x)$ is analytic at infinity. His method of proof however did not involve polynomial sets.

In discussing the equation

$$(16) \quad L[y(x)] \equiv y(x+1) - y(x) = F(x),$$

a particular case of (11), A. Hurwitz [7] used the set $\{Q_n\}$, which for (16) is the set of Bernoulli polynomials. Since series in these polynomials serve to define only a limited class of entire functions, whereas he was interested in the case where $F(x)$ is an arbitrary entire function, Hurwitz modified the polynomials by adding to them linear combinations of properly chosen exponential functions. The device used has been termed the *method of expanding contours*. For from the generating relation

$$(17) \quad \frac{te^{tx}}{e^t - 1} = \sum_0^{\infty} Q_n(x) t^n$$

for the Bernoulli set, valid for $|t| < 2\pi$, follows the formula

$$(18) \quad Q_n(x) = \frac{1}{2\pi i} \int_C \frac{e^{tx} dt}{(e^t - 1)t^{n+1}},$$

C being a contour around $t=0$ and lying in $|t| < 2\pi$. If now C is expanded so as to include a certain number of poles of the integrand, the

⁸ A definition will be given later.

new contour, call it C_n , will define for each $n=0, 1, \dots$ a function

$$(19) \quad Q_n^*(x) = \frac{1}{2\pi i} \int_{C_n} \frac{e^{tz} dt}{(e^t - 1)t^n},$$

which is $Q_n(x)$ augmented by certain exponentials (the residues at the poles lying within C_n). Hurwitz proved that to every entire function $F(x)$ given by (1) ($a=0$) there is an entire function solution of (16) given by (15) with Q_n replaced by Q_n^* .

Carmichael [4] extended the method and the conclusion to equation (11). The method of expanding contours can be shown to apply to (11) and (16) even when $F(x)$ is not an entire function. All that is required is that $F(x)$ be analytic about some point in a circle of radius exceeding a number r , whose (fixed) value is determined by the equation and the method. That is, one obtains what may be called a *semi-local* solution rather than a completely local one.

This restrictive condition on $F(x)$ can be, and is, lifted by emphasizing the set $\{P_n\}$ rather than $\{Q_n\}$ [12], [13]. For equation (11), $P_n(x)$ is given by

$$(20) \quad P_n(x) = \frac{1}{n!} [a_1(x + \omega_1)^n + \dots + a_k(x + \omega_k)^n].$$

Let the points $-\omega_j, j=1, \dots, k$, be plotted. For each x let

$$\rho_j(x) = |x + \omega_j|, \quad \rho(x) = \max \{\rho_1(x), \dots, \rho_k(x)\}.$$

As x varies, there is a unique point x^* at which $\rho(x)$ attains its minimum value $\rho = \rho^*$. This means that the circle of radius ρ^* and center x^* will cover the set $\{-\omega_j\}$, and that no other circle of equal or smaller radius will have this property.

Now consider (20). It is undesirable, from the point of view of convergence, to consider those values of x for which the maximum $\rho(x)$ is attained by two or more of the $\rho_j(x)$'s. The set of such points x , which we shall term the *critical set*, is to be excepted. The following results can be proved: *If the series $\sum n! c_n P_n(x)$ converges at $x=x_0$ (not in the critical set), then it converges absolutely in the circular polygon*

$$(21) \quad \rho_j(x) = |x + \omega_j| < \rho(x_0), \quad j = 1, \dots, k,$$

and uniformly in every closed region therein; and a necessary and sufficient condition that the above series converge for at least one x not in the critical set is that $\limsup |c_n|^{1/n} < 1/\rho^*$.

When such a series converges, and thus converges uniformly in some region, the function defined by the series is of course analytic.

The converse is also true: *If and only if a function $F(x)$ is analytic at $x=x^*$ can it be expressed as a convergent P_n -series.* The manner of proof is roundabout. Set $F(x)=1/(x-\alpha)$ in (11), α being a parameter. The resulting equation

$$(22) \quad L[W(x; \alpha)] = \frac{1}{x - \alpha}$$

is shown to have two (or three, according to the case) meromorphic solutions, from which follow two (or three) P_n -expansions for $1/(x-\alpha)$, valid in some neighborhood of $x=x^*$, for α on two (or three) arcs that form a closed contour around x^* . Application of the Cauchy integral formula then shows that $F(x)$, analytic at x^* , also has a convergent P_n -series. It follows, also, that *if $F(x)$ is analytic about x^* there is an analytic solution of (11) in the neighborhood of x^** ; and a translation of the variable x permits the point x^* to be replaced by any other point of the plane.

It should be added that Appell expansions are usually *not* unique; that is, that the function zero possesses at least one P_n -expansion in which the coefficients are not all zero. To see this we turn back to (5). If $t_0 \neq 0$ is a zero of the generating function $A(t)$, then

$$(23) \quad 0 = \sum_0^{\infty} t_0^n P(x)$$

is an expansion of zero.

We now consider the more general functional equation

$$(24) \quad A[y(x)] \equiv \sum_{k=0}^{\infty} a_k y^{(k)}(x) = F(x),$$

which is a linear differential equation of infinite order, with constant coefficients. Equation (11) is subsumed under (24). For if we call

$$(25) \quad A(t) = \sum_0^{\infty} a_k t^k$$

the *generating function for operator $A[y]$* , then (11) is that case of (24) for which

$$(26) \quad A(t) = \sum_{j=1}^k a_j e^{a_j t}.$$

The Appell set $\{P_n\}$ defined by (12) likewise has as its generating function the $A(t)$ of (25). Now the expansion problem will differ with the character of $A(t)$. In general, the larger the class of functions that

we wish to expand in a $\{P_n\}$ -series, the greater must be the restriction on $A(t)$. A fruitful assumption, and one that we now make, is that $A(t)$ is of *finite exponential type*.⁹ That is, the coefficients of (25) satisfy the inequality

$$(27) \quad \limsup_{n \rightarrow \infty} |n!a_n|^{1/n} < \infty.$$

If this superior limit has the value σ , then $A(t)$ is said to be of *exponential type σ* (exp. type σ).

From (6) we see that $A(t)e^{tx}$ is, in the variable t , of exp. type not exceeding $\sigma + |x|$, so that $\limsup |n!P_n(x)|^{1/n} \leq \sigma + |x|$. This is, however, too crude an estimate. To better it, we observe that if a function $C(t) = \sum_0^\infty c_n t^n$ is of exp. type σ , then it is the *associated entire function* (AEF) relative to the function $C_1(t) = \sum_0^\infty n!c_n t^n$; and this latter function has $1/\sigma$ as its radius of convergence. Conversely, if by studying $C_1(t)$ one can locate the singularity nearest the origin, that will serve to determine the type of $C(t)$.

Applying this principle to (5), it is found [14] on setting

$$(28) \quad A(t) = \text{AEF}\{A_1(t)\}, \quad A^*(t) = tA_1(t),$$

that

$$(29) \quad A^*\left(\frac{1}{t-x}\right) = \sum_0^\infty \frac{n!P_n(x)}{t^{n+1}},$$

$|t|$ sufficiently large. Let $G = \{\alpha\}$ be the set of all the singularities of $A^*(t)$ in the whole plane, and define the continuous function $D(x)$ by

$$(30) \quad D(x) = \max_G |x + 1/\alpha|.$$

It can then be established that

$$(31) \quad \limsup |n!P_n(x)|^{1/n} = D(x).$$

Let us call the curves $D(x) = c$ the *level curves* for the set $\{P_n\}$. A point set discussion reveals the following information: There is a unique point x^* where $D(x)$ has its minimum value D_m . For every $c > D_m$, the level curve is a *simple closed convex curve*, containing in its interior the level curves $D(x) = c'$ for every c' in $D_m \leq c' < c$. In the case that the points $\{\alpha\}$ are finite in number, the level curves consist of circular arcs. This is the situation relative to equation (11), where

⁹ See Muggli [9], where other cases are also taken up. The solution of (24) is there carried out by the method of expanding contours, and for the case where $A(t)$ is of finite exponential type, $F(x)$ is assumed to be an entire function.

the region of convergence of a P_n -expansion was found to be a circular polygon.

With respect to the present general case we find that: *If $\limsup |c_n|^{1/n} = \rho < 1/D_n$, then series $\sum n!c_nP_n(x)$ converges absolutely interior to the level curve $D(x)=1/\rho$, and converges uniformly in every closed region therein, thus representing an analytic function there.*

At this point we are forced to abandon the method that proved so useful in dealing with (11), for we have not sufficient knowledge of solutions of the equation corresponding to (22) to follow the earlier plan. Turning to the polynomial set $\{Q_n\}$ of (12), and modifying it by the method of expanding contours, it is possible, first to solve equation (24) *semi-locally*, and then to obtain a P_n -expansion theorem: *If $F(x)$ is analytic about a point $x=x_0$ in a circle of radius exceeding λ , where λ is determined by the operator A , then equation (24) has an analytic solution $y(x)$ in some neighborhood of x_0 ; and if $F(x)$ is analytic about $x=0$ in a circle of radius exceeding λ^* (again determined by operator A), then $F(x)$ has a convergent Appell expansion*

$$(32) \quad F(x) = \sum_0^{\infty} n!c_nP_n(x).$$

The numbers λ, λ^* are not sharply defined. Burdette [3] has obtained sharp values on making the additional hypothesis that $A(t)$ is bounded on a certain infinite sequence of expanding contours.

3. Sets of type zero. Let us turn back to equation (16). Relative to it one can consider the set of Newton polynomials

$$(33) \quad N_n(x) = \frac{x(x-1) \cdots (x-n+1)}{n!};$$

for this set, which is not an Appell set, has the fundamental property of reproducing itself under application of the difference operator:

$$(34) \quad \Delta N_n(x) = N_{n-1}(x),$$

so that if $F(x)$ has the expansion

$$(35) \quad F(x) = \sum_0^{\infty} c_n N_n(x),$$

then a formal solution of (16) is given by

$$(36) \quad y(x) = \sum_0^{\infty} c_n N_{n+1}(x).$$

We may say that $\{N_n\}$ is a set of *difference polynomials*, as is every other set (and there are infinitely many such) that has the property (34). Such sets may be examined as were Appell sets, but at this point there is suggested a general problem: Can there be assigned to an arbitrary set $\{P_n\}$ an operator J that will reproduce $\{P_n\}$ in the sense that

$$(37) \quad J[P_n(x)] = P_{n-1}(x)$$

for all n ?

The answer is in the affirmative, and J can be expressed in the form

$$(38) \quad J[y] = \sum_{n=1}^{\infty} L_{n-1}(x) y^{(n)}(x)$$

where $L_k(x)$ is a polynomial of degree not exceeding k . This too is a universal (or canonical) operator, since to every set corresponds a determined J . For Appell sets and for difference sets, the L_n 's are constants. This suggests examination of all sets with constant L_n 's. We shall say that a set $\{P_n\}$ is of *zero type* [15] if in its associated operator J (as given by (38)) the polynomials $L_n(x)$ are all constants.

For zero type sets, then, (38) can be written

$$(39) \quad J[y(x)] = \sum_{n=1}^{\infty} c_n y^{(n)}(x), \quad c_1 \neq 0.$$

Let the formal power series

$$(40) \quad J(t) = \sum_1^{\infty} c_n t^n$$

be called the *generating function* for (39), and let $H(t)$ be the inverse power series to $J(t)$, so that (formally)

$$(41) \quad J(H(t)) = H(J(t)) = t.$$

Zero type sets can then be characterized as follows: $\{P_n\}$ is a set of type zero if and only if a formal power series $A(t) = \sum_0^{\infty} a_n t^n$ exists so that

$$(42) \quad A(t) e^{xH(t)} = \sum_0^{\infty} P_n(x) t^n.$$

Observe that the Laguerre polynomials $\{L_n(x)\}$ are of zero type, since

$$(43) \quad \frac{1}{1-t} \exp\left(\frac{-xt}{1-t}\right) = \sum_0^{\infty} L_n(x) t^n.$$

Various other characterizations of zero type sets can be found. One of the simplest is that *constants* $\{q_{k0}\}, \{q_{k1}\}$ exist so that the recurrence relation

$$(44) \quad \sum_{k=1}^{\infty} (q_{k0} + xq_{k1})P_{n-k}(x) = nP_n(x)$$

holds for $n = 1, 2, \dots$. The relation between (42) and (44) is expressed by the series

$$(45) \quad \frac{A'(t)}{A(t)} = \sum_0^{\infty} q_{n+1,0}t^n, \quad H'(t) = \sum_0^{\infty} q_{n+1,1}t^n.$$

Having learned that among Appell sets is a Tchebycheff set (the Hermite polynomials) one naturally asks if there are other orthogonal sets in the extended class of zero type sets. This problem was solved by Meixner [8], by use of the Laplace transformation. He took (42) as his definition of the polynomials that he was investigating. Following Meixner, Geronimus obtained some interesting properties of zero type sets. A second resolution of the Meixner problem can be made by the combined use of (44) and the characterizing recurrence relation

$$(46) \quad Q_n(x) = (x + \lambda_n)Q_{n-1}(x) + \mu_n Q_{n-2}(x), \quad n = 1, 2, \dots,$$

for orthogonal polynomials. For (46) to represent a set of type zero it is necessary and sufficient that

$$(47) \quad \lambda_n = a + bn, \quad \mu_n = (n-1)(c + dn),$$

with $c + dn \neq 0$ for $n > 1$. From this will follow the explicit determination of the required sets, as (otherwise) obtained by Meixner.

Consider an operator J of form (39). The equation (24) can be equally well expressed in terms of J :

$$(48) \quad A[y(x)] = \sum_{k=0}^{\infty} a_k^* J^k[y(x)] = \bar{F}(x),$$

where $J^k[y] \equiv J[J^{k-1}[y]]$, $J^0[y] \equiv y$. As such, we see that polynomial sets of zero type suggest themselves as a means of studying equation (24). This study, however, awaits more knowledge of the expansion properties of such sets.

4. **On equations with non-constant coefficients.** The functional equations already considered have been of the constant coefficient type, and it may be asked if zero type sets can be applied to more general equations. Such application has been made to the linear differ-

ential equation of finite order, and this points the way to other equations (where, however, the results are as yet only formal).

Consider the differential equation

$$(49) \quad L[y(x)] \equiv y^{(k)} + \rho_1(x)y^{(k-1)} + \cdots + \rho_k(x)y = F(x).$$

We seek [16] an operator $M[y]$ that is inverse to $L[y]$ in accordance with the relations

$$(50) \quad ML[y] \equiv y; \quad LM[y] \equiv y.$$

The assumed form of $M[y]$ is

$$(51) \quad M[y] \equiv \sum_{n=-\infty}^{+\infty} M_n(x)y^{(n)}(x),$$

where

$$(52) \quad y^{(-n)}(x) = \int_{x_0}^x \cdots \int_{x_0}^x y(t) dt, \quad n = 1, 2, \dots$$

(n-fold)

It is found that (50) can be satisfied by choosing $M_n(x) = 0$, $n > -k$; $M_{-k}(x) = 1$; and the other M 's by recurrence:

$$(53) \quad M_{-m}(x) = C_{m,m-1}(x)M_{-(m-1)}(x) + C_{m,m-2}(x)M_{-(m-2)}(x) + \cdots \\ + C_{m,k}(x)M_{-k}(x), \quad m > k.$$

(The functions $C_{ij}(x)$ are determined by the coefficients $\rho_n(x)$ and their derivatives.) On setting

$$(54) \quad H(x, t) = M_{-k}(x) \frac{(x-t)^{k-1}}{(k-1)!} + M_{-(k+1)}(x) \frac{(x-t)^k}{k!} + \cdots,$$

a solution of (49) is given by

$$(55) \quad y(x) = \int_{x_0}^x H(x, t)F(t)dt.$$

$H(x, t)$ itself satisfies (in the variable x) the homogeneous equation

$$(56) \quad L[u] = 0,$$

as do also $\partial H/\partial t$, $\partial^2 H/\partial t^2$, $\dots$, $\partial^{k-1} H/\partial t^{k-1}$; and these k functions form an independent set of solutions. •

The series (54) has an interesting domain of convergence, but we must forego a discussion of this [16]. What is of importance for our present aim is the observation that the Appell polynomial set $\{(x-t)^n/n!\}$ is present (at least, beginning with $n=k-1$) in this

series. Any other Appell set would (formally at least) lead to the same result, in the following sense: If $\{P_n(x)\}$ is an Appell set then the formal series

$$(57) \quad K(x, t) = M_{-k}(x)P_{k-1}(x-t) + M_{-(k+1)}(x)P_k(x-t) + \dots$$

satisfies (56). The convergence problem for (57) has not been examined, although relation (31) is certainly applicable.

With (49) as a guide, we turn to the corresponding difference equation

$$(58) \quad L[u] \equiv \Delta^k u(x) + q_1(x)\Delta^{k-1}u(x) + \dots + q_k(x)u(x) = F(x).$$

Here we look for an inverse operator $M(LM[u] \equiv ML[u] \equiv u)$ in the form

$$(59) \quad M[u] \equiv \sum_{n=-\infty}^{+\infty} M_n(x)\Delta^n u(x),$$

where for negative n the "sum" is meant; that is, $\Delta^n[\Delta^{-n}[u]] = u$. We can again choose $M_n(x) = 0$ for $n > -k$; $M_{-k}(x) = 1$; after which $M_{-(k+1)}(x)$, $M_{-(k+2)}(x)$, $\dots$ are determined by recurrence, in analogy with (53). And like (54) (or (57)) we set up the (formal) series

$$(60) \quad K(x, t) = M_{-k}(x)P_{k-1}(x-t) + M_{-(k+1)}(x)P_k(x-t) + \dots$$

where $\{P_n\}$ is any difference set ($\Delta P_n = P_{n-1}$).

Convergence has not yet been examined. Formally, however, (60) satisfies in the variable x the homogeneous equation $L[u] = 0$, as do the functions ΔK , $\Delta^2 K$, $\dots$, where the differences are taken with respect to the variable t .

5. Remarks on null-functions. We shall close with some remarks on the problem of null-functions (as we may call them). Let an infinite sequence of functionals $\{\mathcal{M}_n\}$ be given. We assume them to be of the form

$$(61) \quad \mathcal{M}_n[f(x)] = c_{n0}f(0) + c_{n1}f'(0) + c_{n2}f''(0) + \dots, \quad n = 0, 1, \dots,$$

supposed applicable to a given class of analytic functions. By a *null-function* we shall understand a function $f(x)$, not identically zero, for which

$$(62) \quad \mathcal{M}_n[f(x)] = 0, \quad n = 0, 1, \dots$$

The determination of null-functions is linked to the problem of finding the *radius of completeness* of the sequence of functions $\{M_n(t)\}$, where

$$(63) \quad M_n(t) = c_{n0} + c_{n1}t + c_{n2}t^2 + \cdots;$$

that is, the largest number r such that every function $f(t)$, analytic in $|t| < r$, has an expansion

$$(64) \quad f(t) = \sum_0^{\infty} f_n M_n(t)$$

convergent in $|t| < r$ and uniformly convergent in every closed region therein.¹⁰

An interesting subclass of (61) is that in which $\mathcal{M}_n$ has the form

$$(65) \quad \mathcal{M}_n[f] = c_{nn}f^{(n)}(0) + c_{n,n+1}f^{(n+1)}(0) + \cdots, \quad c_{nn} \neq 0.$$

An example of this, namely when

$$(66) \quad \mathcal{M}_n[f] = f^{(n)}(a_n), \quad |a_n| \leq 1, \quad n = 0, 1, \cdots,$$

leads to the Takenaka problem [18] of determining the largest number λ for which it is true that no entire function of exp. type less than λ can be a null-function. It is known that $\lambda \geq \log 2$, and it is believed that $\lambda = \pi/4$. The example

$$f(x) = \cos \frac{\pi x}{4} - \sin \frac{\pi x}{4}, \quad a_n = (-1)^n,$$

shows that $\pi/4$ cannot be exceeded.

Corresponding to the above sequence $a_n = (-1)^n$ (and it may very well be true for every sequence for which $\{a_n\}$ is bounded), the following is true: Given any $\sigma > 0$, there is only a *finite number* of linearly independent null-functions of exp. type not exceeding σ . (As $\sigma \rightarrow \infty$, however, the number of independent null-functions is unbounded.) If, however, the sequence $\{a_n\}$ is not required to be bounded, and in other cases of the functionals $\{\mathcal{M}_n\}$, there may be a continuous array of null-functions. This fact we shall illustrate by borrowing from the theory of sets of type zero. Consider, in fact, the set $\{P_n\}$ for which

$$(67) \quad e^{zH(t)} = \sum_0^{\infty} P_n(x)t^n;$$

or, letting $J(t)$ be the inverse of $H(t)$,

$$(68) \quad e^{zt} = \sum_0^{\infty} P_n(x)[J(t)]^n.$$

¹⁰ For work in this direction, consult Whittaker [18], Gontcharoff [6] and Boas [2]. These works include further references.

Take $[J(t)]^*$ as the function $M_n(t)$ of (63), which thus determines the functionals $\mathcal{M}_n[f]$.

A simple example is given by $H(t) = t = J(t)$, for then $\mathcal{M}_n[f] = f^{(n)}(0)$, so that there is no null-function that is analytic at the origin. Now take $H(t) = \log(t+1)$, $J(t) = e^t - 1$. One verifies that the function $f(x) = e^{\alpha x}(1 - e^{2k\pi ix})$ is a null-function for every integer k and for every value of α . That is, there is a continuous spectrum of null-functions. This property is shared by functionals $\{\mathcal{M}_n\}$ corresponding to other choices of $H(t)$.

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ON TRIGONOMETRICAL SERIES WHOSE COEFFICIENTS DO NOT TEND TO ZERO¹

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Let $\sum \rho_n \cos (nx - \alpha_n)$ be a series (S) such that

$$(1) \quad \rho_n \geq 0, \quad \limsup \rho_n > 0.$$

It is well known that the set E of convergence of S is of measure zero, a result due to Cantor and Lebesgue. More recently Rajchman has proved that E is a sum of an enumerable sequence of H -sets, and so in particular (an H -set being closed and of measure zero) that E is of the first category.²

We propose to establish a more precise property of the sets E and to show the connection between these sets and sets of absolute convergence (that is, sets in which a trigonometrical series can converge absolutely without being absolutely convergent everywhere).

We propose to call, in memory of Rajchman, "set of the type R " any set E such that a series (S) exists which satisfies the condition (1) and converges in E . E being a sum of closed sets it is natural to investigate the properties of perfect sets of the type R . Let P be such a set. For every x belonging to P we must have $\lim \rho_n \cos (nx - \alpha_n) = 0$. But by (1) there exists an infinite sequence of integers $\{n_k\}$ such that $\rho_{n_k} > r > 0$. Hence,

$$(2) \quad \lim \cos (n_k x - \alpha_{n_k}) = 0.$$

We can assume, without loss of generality that the point $x=0$ belongs to P . Hence

$$(3) \quad \lim \cos \alpha_{n_k} = 0.$$

(2) and (3) give immediately

$$(4) \quad \lim |\sin n_k x \sin \alpha_{n_k}| = 0,$$

and as by (3)

$$(5) \quad \lim |\sin n_k x \cos \alpha_{n_k}| = 0$$

we get, by adding (4) and (5),

$$(6) \quad \lim |\sin n_k x| = 0.$$

¹ Presented to the Society, May 2, 1941.

² See Zygmund, *Trigonometrical Series*, Warsaw, 1935, pp. 267-270.

This implies, for every x belonging to P , $\lim \sin^2 n_k x = 0$. Let $F(x)$ be any non-decreasing bounded function, constant in each interval contiguous to P , but not everywhere. The sequence $\{\sin^2 n_k x\}$ being uniformly bounded, we have

$$\lim \int_0^{2\pi} \sin^2 n_k x dF = 0$$

or

$$(7) \quad \lim \int_0^{2\pi} \cos 2n_k x dF = F(2\pi) - F(0),$$

that is to say, not only the Fourier-Stieltjes cosine coefficients of rank $2n_k$ of dF do not tend to zero (even if F is continuous), but they tend to the greatest possible limit as $k \rightarrow \infty$. It is interesting to observe that the sequence $\{n_k\}$ is independent of the function F .

It has been proved³ that if a perfect set P is a set of absolute convergence, then for every function F of the above-described type

$$\limsup \int_0^{2\pi} \cos 2nx dF = F(2\pi) - F(0).$$

Thus it appears that sets of the type R and sets of absolute convergence are closely connected.

The necessary condition (7) shows that the condition that a set of the type R should be a sum of H -sets is too lax. In particular, even a single H -set is not, in general, of the type R , as can be seen by the example of Cantor's ternary set C . This is an easy consequence of the necessary condition (6) and of the well known fact that the set of all possible sums $x+y$ where x and y belong to C fills the whole interval on which C is constructed.⁴

Moreover, the sum of two perfect sets of the type R may not be of the type R . This result which has been proved by Marcinkiewicz⁵ for perfect sets of absolute convergence holds good for sets of the type R and is a consequence of Marcinkiewicz's argument and of the necessary condition (6).

Let now P be a perfect set such that the condition (7) is satisfied. We can find a function F , bounded, constant in every interval con-

³ See Salem, *The absolute convergence of trigonometrical series*, Duke Mathematical Journal, vol. 8 (1941), pp. 317-334.

⁴ See Zygmund, loc. cit., p. 133.

⁵ Travaux de la Société des Sciences et des Lettres de Wilno, Classe des Sciences Mathématiques et Naturelles, vol. 12 (1937).

tiguous to P , but increasing from one interval to another, and a sequence $\{m_k\}$ such that $\int_0^{2\pi} \sin^2 m_k x dF < 1/k^4$. Hence

$$\sum_1^\infty k^2 \int_0^{2\pi} \sin^2 m_k x dF < \infty.$$

Hence "almost everywhere" in P , that is to say, in a subset P_1 of P such that the variation of F over $P - P_1$ is zero, $\sum_1^\infty k^2 \sin^2 m_k x < \infty$. But

$$\sum_1^k |\sin m_k x| < \left(\sum_1^k \frac{1}{k^2} \right)^{1/2} \left(\sum_1^k k^2 \sin^2 m_k x \right)^{1/2};$$

hence $\sum |\sin m_k x|$ converges in P_1 , and P is "almost everywhere" of the type R (and also, almost everywhere, a set of absolute convergence).

MONTRÉAL, CANADA

UNITARY SPACES WITH CORRESPONDING GEODESICS¹

N. COBURN

1. **Introduction.** This paper is divided into three parts. In the first section, the notation and fundamental concepts of hermitian geometry are reviewed. The second section develops the equations of geodesic curves X_1 which depend on a *real parameter* (t) and which are imbedded in a unitary space of n -dimensions K_n . Our principal result is: The equations of such geodesics differ from the equations of geodesics in Riemannian space in that the former contain the torsion affnor. In the third section, we classify the connections of two unitary spaces $K_n, 'K_n$ whose geodesics correspond. First, we find the necessary and sufficient conditions that two unitary spaces $K_n, 'K_n$ both with symmetric connection shall have their geodesics in correspondence. This last problem is solved in exactly the same manner as the similar problem in Riemannian space.² Secondly, we prove that if K_n has torsion and $'K_n$ has no torsion (symmetric connection), then their geodesics do not correspond. The problem of determining all connections of unitary spaces $K_n, 'K_n$ both with torsion whose geodesics correspond is left open.

¹ Presented to the Society, January 1, 1941.

² L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1926, p. 131.

2. **Notation.**³ Consider a real space of $2n$ -dimensions X_{2n} whose coordinates are given by the real variables

$$(2.1) \quad x^\lambda, y^\lambda, \quad \lambda, \mu = 1 \cdots n.$$

Into this X_{2n} , we introduce the complex coordinates given by

$$(2.2) \quad \begin{aligned} \xi^\lambda &= x^\lambda + iy^\lambda, \\ \xi^{\lambda*} &= x^\lambda - iy^\lambda, \end{aligned} \quad i = (-1)^{1/2}.$$

Since the Jacobian of this transformation $(-2i)$ does not vanish over X_{2n} , then $\xi^\lambda, \xi^{\lambda*}$ constitute a set of $2n$ independent variables which map the X_{2n} . In view of the fact that $\xi^{\lambda*}$ are complex conjugates to ξ^λ , we determine the points of X_{2n} by merely assigning complex numbers to ξ^λ . We say that the ξ^λ determine "points" which build a *complex* space of n -dimensions X_n (the above real topological X_{2n}). Let us denote partial derivatives by

$$(2.3) \quad \partial_\mu = \partial/\partial\xi^\mu, \quad \partial_{\mu*} = \partial/\partial\xi^{\mu*},$$

and let $\phi(\xi^\lambda, \xi^{\lambda*})$ be an analytic function of the variables $\xi^\lambda, \xi^{\lambda*}$. Then, by the composite function theorem, we obtain

$$(2.4) \quad \partial\phi/\partial x^\mu = \partial_\mu\phi + \partial_{\mu*}\phi,$$

$$(2.5) \quad \partial\phi/\partial y^\mu = i\partial_\mu\phi - i\partial_{\mu*}\phi.$$

Solving for $\partial_\mu\phi, \partial_{\mu*}\phi$, we find

$$(2.6) \quad \partial_\mu\phi = 1/2(\partial\phi/\partial x^\mu - i\partial\phi/\partial y^\mu),$$

$$(2.7) \quad \partial_{\mu*}\phi = 1/2(\partial\phi/\partial x^\mu + i\partial\phi/\partial y^\mu).$$

If the function $\psi(\xi^\lambda)$,

$$(2.8) \quad \psi(\xi^\lambda) = u(x^\lambda, y^\lambda) + iv(x^\lambda, y^\lambda),$$

is analytic in the sense of Cauchy-Riemann, then

$$(2.9) \quad \partial u/\partial x^\lambda - \partial v/\partial y^\lambda = 0,$$

$$\partial v/\partial x^\lambda + \partial u/\partial y^\lambda = 0.$$

By expanding the right-hand side of (2.7), we find that (2.9) is equivalent to

$$(2.10) \quad \partial_\lambda\psi = 0.$$

From our point of view, this equation follows directly from the fact

³ J. A. Schouten and D. J. Struik, *Einführung in die Neueren Methoden der Differentialgeometrie*, P. Noordhoff, Groningen, Batavia, vol. 2, 1938, p. 225.

that ξ^λ , $\xi^{\lambda*}$ are independent variables. Hence equation (2.9) merely serves to interpret the equation (2.10) from the point of view of complex variable. If ψ^* is the complex-conjugate function to ψ ,

$$(2.11) \quad \psi^* = u - iv,$$

then it is easily shown that (2.9) is equivalent to the so-called conjugate equation

$$(2.12) \quad \partial_\lambda \psi^* = 0.$$

We seek to generalize the idea involved in (2.12). Corresponding to any function $\phi(\xi^\lambda, \xi^{\lambda*})$, let $\phi^*(\xi^{\lambda*}, \xi^\lambda)$ denote the function obtained by replacing i by $-i$. From (2.2), we see that the variables ξ^λ are then replaced by $\xi^{\lambda*}$ and conversely. We call this function ϕ^* , the conjugate of ϕ . In the future, we shall indicate the validity of the conjugate by the abbreviation "conj." It is to be noted on the formal side, that in passing to the conjugate, all indices will be starred. The star of a starred quantity removes the original star.

Consider the allowable analytic coordinate transformations with nonvanishing Jacobian

$$(2.13) \quad \xi^\lambda = \xi^\lambda(\xi^{\lambda'}), \quad \xi^{\lambda'} = x^{\lambda'} + iy^{\lambda'}.$$

The corresponding conjugate equations are

$$(2.14) \quad \xi^{\lambda*} = \xi^{\lambda*}(\xi^{\lambda''}), \quad \xi^{\lambda''} = x^{\lambda''} - iy^{\lambda''}.$$

We now introduce the unit affinor whose intermediary components are

$$(2.15) \quad A_{\mu'}^\lambda = \partial \xi^\lambda / \partial \xi^{\mu'}, \quad A_{\mu''}^{\lambda*} = \partial \xi^{\lambda*} / \partial \xi^{\mu''}.$$

A vector $v^\lambda(\xi^\lambda, \xi^{\lambda*})$ is said to be of the *first type* if

$$(2.16) \quad v^{\lambda'} = A_{\lambda}^{\lambda'} v^\lambda.$$

Associated with each such vector is its conjugate, or vector of the *second type* $v^{\lambda*}$, with transformation law

$$(2.17) \quad v^{\lambda''} = A_{\lambda}^{\lambda'' \lambda*} v^{\lambda*}.$$

The theory can be extended to affinors of any mixed valence.

Let us introduce a connection in X_n by means of the n^2 quantities $\Gamma_{\mu\alpha}^\lambda$ which are functions of position. Then, we define the covariant derivative of a contravariant vector by

$$(2.18) \quad \delta v^\lambda = dv^\lambda + \Gamma_{\mu\alpha}^\lambda v^\alpha d\xi^\mu, \text{ conj.}$$

The $\Gamma_{\mu\alpha}^\lambda$ transform in a well known manner⁴ under (2.13). We now write for a covariant vector

$$(2.19) \quad \delta w_\lambda = dw_\lambda - \Gamma_{\mu\lambda}^\alpha w_\alpha d\xi^\mu, \text{ conj.}$$

By expanding the ordinary differential of a vector, we obtain

$$(2.20) \quad dv^\lambda = d\xi^\mu \partial_\mu v^\lambda + d\xi^{\mu*} \partial_{\mu*} v^\lambda, \text{ conj.}$$

If we define the covariant derivative of v^λ , w^λ by means of

$$(2.21) \quad \nabla_\mu v^\lambda = \partial_\mu v^\lambda + \Gamma_{\mu\alpha}^\lambda v^\alpha, \quad \nabla_\mu w_\lambda = \partial_\mu w_\lambda - \Gamma_{\mu\lambda}^\alpha w_\alpha,$$

$$(2.22) \quad \nabla_{\mu*} v^\lambda = \partial_{\mu*} v^\lambda, \quad \nabla_{\mu*} w_\lambda = \partial_{\mu*} w_\lambda, \text{ conj.},$$

then

$$(2.23) \quad \delta v^\lambda = d\xi^\mu \nabla_\mu v^\lambda + d\xi^{\mu*} \nabla_{\mu*} v^\lambda, \text{ conj.},$$

$$(2.24) \quad \delta w_\lambda = d\xi^\mu \nabla_\mu w_\lambda + d\xi^{\mu*} \nabla_{\mu*} w_\lambda, \text{ conj.},$$

An hermitian X_n , with covariant derivative defined by (2.21), (2.22) is denoted by K_n .

We now introduce an hermitian tensor with hermitian symmetry

$$(2.25) \quad a_{\lambda\mu*} = (a_{\lambda\mu})^{*'} = a_{\mu* \lambda},$$

the (') indicating the transpose matrix. If we condition $a_{\lambda\mu*}$ by requiring that

$$(2.26) \quad \delta a_{\lambda\mu*} = 0 = (\partial_r a_{\lambda\mu*} - \Gamma_{r\lambda}^\rho a_{\rho\mu*}) d\xi^r + (\partial_{r*} a_{\lambda\mu*} - \Gamma_{r*\mu}^\rho a_{\lambda\rho*}) d\xi^{r*},$$

then the space K_n is said to be a unitary K_n . For such a space, from (2.26), we can prove⁵

$$(2.27) \quad \nabla_r a_{\lambda\mu*} = \partial_r a_{\lambda\mu*} - \Gamma_{r\lambda}^\rho a_{\rho\mu*} = 0, \text{ conj.},$$

$$(2.28) \quad \nabla_{r*} a_{\lambda\mu*} = \partial_{r*} a_{\lambda\mu*} - \Gamma_{r*\mu}^\rho a_{\lambda\rho*} = 0, \text{ conj.}$$

The $a_{\lambda\mu*}$ is now a fundamental tensor and can be used to raise and lower indices through the ∇ operator. If we define the contravariant fundamental tensor $a^{\sigma* \kappa}$ by

$$(2.29) \quad a^{\sigma* \kappa} a_{\lambda\sigma*} = A_\lambda^\kappa, \text{ conj.},$$

then (2.27), (2.28) may be solved for the connection

$$(2.30) \quad \Gamma_{\mu\lambda}^\kappa = (\partial_\mu a_{\lambda\sigma*}) a^{\sigma* \kappa} = (\partial_\mu a_{\sigma* \lambda}) a^{\sigma* \kappa},$$

$$(2.31) \quad \Gamma_{\mu* \lambda}^{\kappa*} = (\partial_{\mu*} a_{\sigma* \lambda}) a^{\sigma* \kappa} = (\partial_{\mu*} a_{\lambda\sigma*}) a^{\sigma* \kappa}.$$

⁴ J. A. Schouten and D. J. Struik, loc. cit., p. 227.

⁵ J. A. Schouten and D. J. Struik, loc. cit., p. 234.

Finally, we introduce the torsion affinor

$$(2.32) \quad S_{\mu\lambda}^{\cdot\cdot\cdot\epsilon} = 1/2(\Gamma_{\mu\lambda}^{\epsilon} - \Gamma_{\lambda\mu}^{\epsilon}) = \Gamma_{[\mu\lambda]}^{\epsilon}, \text{ conj.}$$

The sign [] means that the antisymmetric product of the enclosed indices is to be formed; the sign () means that the symmetric product of the enclosed indices is to be formed.

3. Equations of geodesics in K_n . Consider the curve X_1 defined in K_n by means of the equations

$$(3.1) \quad \xi^\lambda = \xi^\lambda(t), \quad \xi^{\lambda*} = \xi^{\lambda*}(t),$$

where (t) is a real parameter and $\xi^\lambda, \xi^{\lambda*}$ are conjugate functions. Evidently, such equations determine a curve X_1 in the real topological X_{2n} . We define the element of arc length along this X_1 by

$$(3.2) \quad ds = (a_{\lambda\mu} d\xi^\lambda d\xi^{\mu*})^{1/2} dt.$$

By substituting (3.1) into (3.2) and integrating, we obtain (s) the arc length parameter. It is to be noted that this parameter exists only because of the fact that (t) is real. By use of (2.25) and the conjugate of (3.2), we see that (ds) and hence (s) are real.

We follow the well known methods of the calculus of variations⁶ in finding the geodesics of K_n . Our problem is to find that X_1 for which the first variation of

$$(3.3) \quad I = \int_{t_0}^{t_1} (a_{\lambda\mu} d\xi^\lambda d\xi^{\mu*})^{1/2} dt$$

is zero. Let ξ^λ represent the coordinates of a point on the geodesic X_1 ; let ξ'^λ represent the coordinates of a point on the varied X_1 ; let ϵ be an infinitesimal; and let ω^λ be an arbitrary function of $\xi^\lambda, \xi^{\lambda*}$ which vanishes at $t=t_0, t_1$. The equations of variation can be written as

$$(3.4) \quad \xi'^\lambda = \xi^\lambda + \epsilon\omega^\lambda, \text{ conj.}$$

For clarity, we write

$$(3.5) \quad \phi(\xi^\lambda, \xi^{\lambda*}, \xi^\lambda, \xi^{\lambda*}) = (a_{\lambda\mu} d\xi^\lambda d\xi^{\mu*})^{1/2},$$

where

$$(3.6) \quad \xi^\lambda = d_t \xi^\lambda, \quad \xi^{\lambda*} = d_t \xi^{\lambda*}.$$

Then by use of Taylor's series and integration by parts, we obtain for the first variation of (3.3)

⁶ L. P. Eisenhart, loc. cit., p. 49.

$$(3.7) \quad 0 = \int_{t_0}^{t_1} \left[\frac{\partial \phi}{\partial \xi^\lambda} - \frac{d}{dt} \left(\frac{\partial \phi}{\partial \dot{\xi}^\lambda} \right) \right] \omega^\lambda dt + \left[\frac{\partial \phi}{\partial \xi^{\lambda*}} - \frac{d}{dt} \left(\frac{\partial \phi}{\partial \dot{\xi}^{\lambda*}} \right) \right] \omega^{\lambda*} dt.$$

First, let us assume that the vector $(\omega^\lambda, \omega^{\lambda*})$ has the components $(a, b, \dots, a, b, \dots)$ where $a, b, \dots$ are arbitrary real functions of $\xi^\lambda, \xi^{\lambda*}$. Secondly, let us assume that $(\omega^\lambda, \omega^{\lambda*})$ has the components $(ia, ib, \dots, -ia, -ib, \dots)$. Then, by substituting into (3.7), we find that this relation decomposes into the two equations

$$(3.8) \quad 0 = \int_{t_0}^{t_1} \left[\frac{\partial \phi}{\partial \xi^\lambda} - \frac{d}{dt} \left(\frac{\partial \phi}{\partial \dot{\xi}^\lambda} \right) \right] \omega^\lambda dt,$$

$$(3.9) \quad 0 = \int_{t_0}^{t_1} \left[\frac{\partial \phi}{\partial \xi^{\lambda*}} - \frac{d}{dt} \left(\frac{\partial \phi}{\partial \dot{\xi}^{\lambda*}} \right) \right] \omega^{\lambda*} dt.$$

By the ordinary argument, we obtain the Euler equations

$$(3.10) \quad \frac{\partial \phi}{\partial \xi^\lambda} - \frac{d}{dt} \left(\frac{\partial \phi}{\partial \dot{\xi}^\lambda} \right) = 0,$$

$$(3.11) \quad \frac{\partial \phi}{\partial \xi^{\lambda*}} - \frac{d}{dt} \left(\frac{\partial \phi}{\partial \dot{\xi}^{\lambda*}} \right) = 0.$$

We follow Eisenhart in listing the steps of our computation. From (3.5), we have

$$(3.12) \quad \frac{\partial \phi}{\partial \xi^\lambda} = \frac{1}{2} \frac{a_{\lambda\mu} d_i \xi^{\mu*}}{(a_{\lambda\mu} d_i \xi^\lambda d_i \xi^{\mu*})^{1/2}} = \frac{1}{2} \frac{a_{\lambda\mu} d_i \xi^{\mu*}}{(ds/dt)},$$

$$(3.13) \quad \frac{\partial \phi}{\partial \dot{\xi}^\lambda} = \frac{1}{2} \frac{(\partial_\lambda a_{r\mu}) d_i \xi^r d_i \xi^{\mu*}}{(ds/dt)},$$

$$(3.14) \quad \frac{d}{dt} \left(\frac{\partial \phi}{\partial \dot{\xi}^\lambda} \right) = \frac{1}{2} \frac{\frac{ds}{dt} [a_{\lambda\mu} d_i^2 \xi^{\mu*} + \partial_r a_{\lambda\mu} d_i \xi^r d_i \xi^{\mu*} + \partial_r a_{\lambda\mu} d_i \xi^{r*} d_i \xi^{\mu*}] - \frac{d^2 s}{dt^2} (a_{\lambda\mu} d_i \xi^{\mu*})}{(ds/dt)^2}.$$

Substituting these relations into (3.5), we obtain

$$(3.15) \quad a_{\lambda\mu} d_i^2 \xi^{\mu*} + \partial_r a_{\lambda\mu} d_i \xi^r d_i \xi^{\mu*} + \partial_r a_{\lambda\mu} d_i \xi^{r*} d_i \xi^{\mu*} - \partial_\lambda a_{r\mu} d_i \xi^r d_i \xi^{\mu*} - \frac{a_{\lambda\mu} d_i \xi^{\mu*}}{(ds/dt)} \frac{d^2 s}{dt^2} = 0.$$

From (2.30) and (2.31), we find

$$(3.16) \quad \partial_r a_{\lambda\mu} = \Gamma_{r\lambda}^{\kappa} a_{\kappa\mu},$$

$$(3.17) \quad \partial_r^* a_{\lambda\mu} = \Gamma_{r^*\mu}^{*\kappa} a_{\lambda\kappa}.$$

Hence (3.15) becomes

$$(3.18) \quad a_{\lambda\mu} d_i^2 \xi^{\mu} + a_{\kappa\mu} \Gamma_{r\lambda}^{\kappa} d_i \xi^r d_i \xi^{\mu} + \Gamma_{r^*\mu}^{*\kappa} a_{\lambda\kappa} d_i \xi^r d_i \xi^{\mu} - \Gamma_{\lambda\mu}^{\kappa} a_{\kappa\mu} d_i \xi^r d_i \xi^{\mu} - \frac{a_{\lambda\mu} d_i \xi^{\lambda}}{dt^2} \frac{d^2 s}{(ds/dt)} = 0.$$

Transvecting (3.18) with $a^{\alpha\lambda}$, we obtain after combining terms two and four

$$(3.19) \quad d_i^2 \xi^{\alpha} + \Gamma_{r^*\mu}^{*\alpha} d_i \xi^r d_i \xi^{\mu} + 2a_{\kappa\mu} a^{\alpha\lambda} S_{r\lambda}^{\kappa} d_i \xi^r d_i \xi^{\mu} - d_i \xi^{\alpha} \left(\frac{d^2 s}{dt^2} / \frac{ds}{dt} \right) = 0.$$

From the equation (3.11), we obtain the conjugate of (3.19). Hence, we have this theorem:

THEOREM 1. *The geodesic X_1 in unitary K_n which are functions of a real parameter (t) is given by the solutions of the second order differential equation (3.19) and its conjugate.*

Evidently, the equations of the geodesic X_1 in unitary K_n differ from the equations of the geodesic V_1 in V_n in the additional term containing the torsion affnor. It follows that if the unitary space K_n has no torsion, then the geodesic X_1 of K_n satisfies the same type of differential equation as the geodesic V_1 in V_n and conversely.

4. Unitary spaces with corresponding geodesics. By multiplying the conjugate of (3.19) by $d_i \xi^{\beta}$, forming a similar equation with the indices α, β interchanged and then subtracting the two resulting equations, we obtain⁷ for the geodesic X_1 in K_n

$$(4.1) \quad (d_i \xi^{\beta} d_i \xi^{\alpha} - d_i \xi^{\alpha} d_i \xi^{\beta}) + (\Gamma_{r\mu}^{\alpha} d_i \xi^{\beta} - \Gamma_{r\mu}^{\beta} d_i \xi^{\alpha}) d_i \xi^r d_i \xi^{\mu} + 2a_{\mu\kappa} S_{r\lambda}^{*\kappa} (a^{\lambda\alpha} d_i \xi^{\beta} - a^{\lambda\beta} d_i \xi^{\alpha}) d_i \xi^r d_i \xi^{\mu} = 0, \text{ conj.}$$

Consider another n -dimensional unitary space $'K_n$ which is also mapped by the variables ξ^{λ} such that the point $P(\xi^{\lambda})$ of K_n corresponds to the point $'P(\xi^{\lambda})$ of $'K_n$. Furthermore, let the fundamental tensor and connection of this space be denoted by

$$(4.2) \quad 'a_{\lambda\mu}, \quad '\Gamma_{r\mu}^{\kappa}.$$

⁷ L. P. Eisenhart, loc. cit., p. 131.

If the geodesics of $'K_n$ correspond to those of K_n , then equation (4.1) with $a_{\lambda\mu}^\alpha$, $\Gamma_{\lambda\mu}^\alpha$, $S_{r\mu}^{\alpha\beta}$ replaced by $'a_{\lambda\mu}^\alpha$, $'\Gamma_{\lambda\mu}^\alpha$, $'S_{r\mu}^{\alpha\beta}$ is satisfied by the same functions $\xi^\alpha(t)$ which satisfy (4.1), that is,

$$(4.3) \quad (d_i \xi^\beta d_i^2 \xi^\alpha - d_i \xi^\alpha d_i^2 \xi^\beta) + ('\Gamma_{r\mu}^\alpha d_i \xi^\beta - '\Gamma_{r\mu}^\beta d_i \xi^\alpha) d_i \xi^r d_i \xi^\mu + 2 'a_{\mu\kappa}^{\alpha\beta} ('S_{r\lambda}^{\alpha\kappa} d_i \xi^\beta - 'a^{\lambda\beta} d_i \xi^\alpha) d_i \xi^r d_i \xi^\mu = 0, \text{ conj.}$$

Let us first assume that K_n and $'K_n$ both possess symmetric connections.

THEOREM 2. *The necessary and sufficient condition that two unitary spaces K_n , $'K_n$ both with symmetric connections have geodesics in correspondence is that a vector p_r exist such that*

$$' \Gamma_{r\mu}^\alpha = \Gamma_{r\mu}^\alpha + 2p_{(r} A_{\mu)}^\alpha, \text{ conj.}$$

We can write the equation

$$(4.4) \quad '\Gamma_{r\mu}^\alpha = \Gamma_{r\mu}^\alpha + A_{r\mu}^{\alpha\beta}, \text{ conj.,}$$

where $A_{r\mu}^{\alpha\beta}$ is an affnor to be determined. Subtracting (4.1) from (4.3) (note: the torsion affnors are zero in this case), we obtain

$$(4.5) \quad (A_{r\mu}^{\alpha\beta} d_i \xi^\beta - A_{r\mu}^{\beta\alpha} d_i \xi^\alpha) d_i \xi^r d_i \xi^\mu = 0, \text{ conj.}$$

By use of the unit affnor A_γ^α , we may rewrite (4.5) in the simpler form

$$(4.6) \quad (A_{r\mu}^{\alpha\beta} A_\gamma^\beta - A_{r\mu}^{\beta\alpha} A_\gamma^\alpha) d_i \xi^\gamma d_i \xi^r d_i \xi^\mu = 0, \text{ conj.}$$

Since (4.6) is to be satisfied by arbitrary values of $d_i \xi^\alpha$ at any point $P(\xi^\lambda)$ and $'P(\xi^\lambda)$, we find

$$(4.7) \quad A_{(r\mu}^{\alpha\beta} A_{\gamma)}^\beta - A_{(r\mu}^{\beta\alpha} A_{\gamma)}^\alpha = 0, \text{ conj.}$$

Expanding (4.7), we obtain

$$(4.8) \quad A_{(r\mu}^{\alpha\beta} A_{\gamma)}^\beta + A_{(\mu\gamma)}^{\alpha\beta} A_r^\beta + A_{(r\gamma)}^{\alpha\beta} A_\mu^\beta = A_{(r\mu)}^{\beta\alpha} A_\gamma^\alpha + A_{(\mu\gamma)}^{\beta\alpha} A_r^\alpha + A_{(r\gamma)}^{\beta\alpha} A_\mu^\alpha, \text{ conj.}$$

Let us write

$$(4.9) \quad A_{r\alpha}^{\alpha\beta} = A_{\alpha r}^{\alpha\beta} = (n+1)p_r, \text{ conj.,}$$

and contract on the indices μ, α in (4.8). Then we obtain

$$(4.10) \quad A_{r\gamma}^{\alpha\beta} = 2p_{(r} A_{\gamma)}^\beta, \text{ conj.}$$

Conversely, if (4.10) is satisfied, then (4.6) is identically satisfied. Hence our theorem is proved.

Let us now assume that K_n has torsion and $'K_n$ has no torsion (symmetric connection).

THEOREM 3. *Two unitary spaces, K_n with torsion, $'K_n$ without torsion, cannot have their geodesics in correspondence.*

Let us assume the contrary, namely, that the geodesics of these two spaces correspond. Then again, we write the equation (4.4) and again, we subtract (4.1) from (4.3). In this case, instead of equation (4.6), we obtain

$$(4.11) \quad (A_{\nu\mu}^{\cdot\cdot\alpha} A_{\gamma}^{\beta} - A_{\nu\mu}^{\cdot\cdot\beta} A_{\gamma}^{\alpha}) d_i \xi^{\gamma} d_i \xi^{\nu} d_i \xi^{\mu} + 2a_{\mu\kappa} S_{\kappa\lambda}^{\cdot\cdot\kappa} (a^{\lambda\alpha} A_{\gamma}^{\beta} - a^{\lambda\beta} A_{\gamma}^{\alpha}) d_i \xi^{\gamma} d_i \xi^{\mu} d_i \xi^{\nu} = 0, \text{ conj.}$$

Equation (4.11) is to be satisfied by arbitrary values of the $d_i \xi^{\alpha}$. Let us first assume that the vector $(d_i \xi^{\alpha}, d_i \xi^{\alpha*})$ has the arbitrary real components $(a, b, c, \dots a, b, c, \dots)$ and secondly, let us assume that this vector has the arbitrary pure imaginary components $(ia, ib, ic, \dots -ia, -ib, \dots)$. By adding the resulting equations, we find that (4.11) decomposes into the two equations

$$(4.12) \quad (A_{\nu\mu}^{\cdot\cdot\alpha} A_{\gamma}^{\beta} - A_{\nu\mu}^{\cdot\cdot\beta} A_{\gamma}^{\alpha}) d_i \xi^{\gamma} d_i \xi^{\nu} d_i \xi^{\mu} = 0, \text{ conj.},$$

$$(4.13) \quad a_{\mu\kappa} S_{\kappa\lambda}^{\cdot\cdot\kappa} (a^{\lambda\alpha} A_{\gamma}^{\beta} - a^{\lambda\beta} A_{\gamma}^{\alpha}) d_i \xi^{\gamma} d_i \xi^{\mu} d_i \xi^{\nu} = 0, \text{ conj.}$$

The equation (4.12) can be analyzed in an exactly analogous manner to that used in the previous theorem excepting that (4.9) must be replaced by

$$(4.14) \quad A_{\nu\alpha}^{\cdot\cdot\alpha} = (n+1)p_{\nu}, \quad A_{\alpha\nu}^{\cdot\cdot\alpha} = (n+1)q_{\nu}, \text{ conj.},$$

since $A_{\nu\mu}^{\cdot\cdot\alpha}$ is no longer symmetric in ν, μ . Instead of (4.10), we find

$$(4.15) \quad A_{(\nu\gamma)}^{\cdot\cdot\beta} = 2r_{(\nu} A_{\gamma)}^{\beta}, \quad 2r_{\nu} = p_{\nu} + q_{\nu}, \text{ conj.}$$

In order to analyze (4.13), we write this equation in the form

$$(4.16) \quad A_{\mu\gamma\rho}^{\alpha\beta} d_i \xi^{\mu} d_i \xi^{\gamma} d_i \xi^{\rho} = 0, \text{ conj.},$$

where

$$(4.17) \quad A_{\mu\gamma\rho}^{\alpha\beta} = a_{\mu\kappa} S_{\kappa\lambda}^{\cdot\cdot\kappa} (a^{\lambda\alpha} A_{\gamma}^{\beta} - a^{\lambda\beta} A_{\gamma}^{\alpha}), \text{ conj.}$$

As previously, let us assume first that $(d_i \xi^{\alpha}, d_i \xi^{\alpha*})$ has the arbitrary real components $(a, b, \dots a, b, \dots)$. Upon substituting into (4.16), we obtain a third degree homogeneous polynomial. We shall explicitly

write those terms which contain a, b ; the remaining terms are formulated in a similar fashion. We have

$$(4.18) \quad A_{111}^{\alpha\beta} a^3 + a^2 b (A_{112}^{\alpha\beta} + A_{121}^{\alpha\beta} + A_{211}^{\alpha\beta}) \\ + ab^2 (A_{122}^{\alpha\beta} + A_{212}^{\alpha\beta} + A_{221}^{\alpha\beta}) + A_{222}^{\alpha\beta} b^3 + \dots = 0.$$

Secondly, let us assume $(d_i \xi^\alpha, d_i \xi^{\alpha*})$ has the arbitrary components $(a, ib, \dots, a, ib, \dots)$. Then (4.16) becomes

$$(4.19) \quad A_{111}^{\alpha\beta} a^3 + a^2 bi (-A_{112}^{\alpha\beta} + A_{121}^{\alpha\beta} + A_{211}^{\alpha\beta}) \\ + ab^2 (A_{122}^{\alpha\beta} + A_{212}^{\alpha\beta} - A_{221}^{\alpha\beta}) - i A_{222}^{\alpha\beta} b^3 + \dots = 0.$$

From these equations, that is, (4.18), (4.19) we conclude

$$(4.20) \quad A_{111}^{\alpha\beta} = A_{222}^{\alpha\beta} = A_{122}^{\alpha\beta} = A_{221}^{\alpha\beta} = 0, \quad A_{(12)1}^{\alpha\beta} = A_{(12)2}^{\alpha\beta} = 0.$$

We can evidently express (4.20) by writing

$$(4.21) \quad A_{(\mu\gamma)r}^{\alpha\beta} = 0, \text{ conj.}$$

By use of (4.17), we find upon expanding

$$(4.22) \quad S_{r,\lambda}^{\cdot\cdot\lambda*} (a_{\mu\kappa}^{\lambda*} A_\gamma^\beta + a_{\gamma\kappa}^{\lambda*} A_\mu^\beta - a_{\mu\kappa}^{\lambda*} A_\gamma^{\lambda\beta} - a_{\gamma\kappa}^{\lambda*} A_\mu^{\lambda\beta}) = 0, \text{ conj.}$$

Placing α equal to μ and contracting, we find

$$(4.23) \quad S_{r,\lambda}^{\cdot\cdot\lambda*} A_\gamma^\beta = n S_{r,\lambda}^{\cdot\cdot\kappa*} a_{\gamma\kappa}^{\lambda\beta}, \text{ conj.}$$

Solving (4.23) for $S_{r,\lambda}^{\cdot\cdot\lambda*}$, we find

$$(4.24) \quad S_{r,\mu}^{\cdot\cdot\alpha*} = S_{r,\mu}^{\cdot\cdot\alpha*}, \text{ conj.,}$$

where

$$(4.25) \quad S_{r,\lambda}^{\cdot\cdot\lambda*} = n S_{r,\lambda}^{\cdot\cdot\lambda*}, \text{ conj.}$$

Since $S_{r,\lambda}^{\cdot\cdot\lambda*}$ is antisymmetric in ν^*, μ^* we have

$$(4.26) \quad S_{(\nu^* \mu^*)}^{\cdot\cdot\alpha*} = 0, \text{ conj.}$$

Contracting on α^*, μ^* we obtain the relation

$$(4.27) \quad S_{r,\cdot} = 0, \text{ conj.}$$

Hence $S_{r,\mu}^{\cdot\cdot\lambda*}$ vanishes. But this is contrary to our assumption. Hence our theorem is proved.

SUR LES SOLUTIONS DES ÉQUATIONS DU TYPE PARABOLIQUE DÉTERMINÉES DANS UNE RÉGION ILLIMITÉE

MIROSLAW KRZYŻAŃSKI

L'un des problèmes relatifs à l'équation de la chaleur consiste en la recherche d'une solution de cette équation, se réduisant à une fonction donnée sur la caractéristique. Cette solution est donnée par l'intégrale classique de Weierstrass.¹ Cependant la question de l'unicité de cette solution n'a été complètement résolue que dans les travaux récentes de M. M. Picone et M. A. Tychonoff.²

Or il y a intérêt à éteindre ces résultats à l'équation linéaire du type parabolique:

$$(1) \quad \sum_{i,k=1}^m A_{ik}(x_1, x_2, \dots, x_m, y) \frac{\partial^2 u}{\partial x_i \partial x_k} = \sum_{j=1}^m a_j(x_1, x_2, \dots, x_m, y) \frac{\partial u}{\partial x_j} + b(x_1, x_2, \dots, x_m, y) \frac{\partial u}{\partial y} + c(x_1, x_2, \dots, x_m, y)u,$$

la forme $\sum_{i,k=1}^m A_{ik} \lambda_i \lambda_k$ étant positive définie, en cherchant la solution u de cette équation déterminée dans une région illimitée $R: 0 \leq y \leq h; -\infty < x_i < +\infty$ ($i=1, 2, \dots, m$), telle que

$$u(x_1, x_2, \dots, x_m, 0) = \phi(x_1, x_2, \dots, x_m).$$

Nous commençons par introduire une classification des fonctions caractérisant la façon dont elles se comportent à l'infini. Convenons d'appeler dans la suite la classe E_α ($\alpha > 0$) une classe de fonctions $F(x)$ qui ont la propriété suivante: il existe deux nombres constants positifs M et K tels que

$$|F(x)| < M e^{K|x|^\alpha}, \quad -\infty < x < +\infty.$$

Nous allons voir que dans la classe E_2 par rapport aux x_i il existe une solution et une seule du problème posé.

Le résultat analogue pour l'équation de la chaleur a été obtenu par

¹ Voir par exemple E. Goursat, *Cours d'Analyse*, Tome 3, Paris, 1927, chap. 29, p. 294.

² P. Picone, *Sul problema della propagazione del calore in un mezzo privo di frontiera*, *Mathematische Annalen*, Tome 101 (1929), pp. 701-712. A. Tychonoff, *Théorèmes d'unicité pour l'équation de chaleur*, *Recueil Mathématique de Moscou*, Tome 42 (1935), pp. 199-214. Cf. aussi le travail récent de M. F. G. Dressel, *The fundamental solution of the parabolic equation*, *Duke Mathematical Journal*, Tome 7 (1940), pp. 186-203.

M. Tychonoff (voir le travail cité), qui a démontré en outre que l'unicité du problème n'a pas lieu dans la classe $E_{2+\epsilon}$, où ϵ est un nombre positif arbitrairement petit.

Considérons l'équation de la forme (1) dont les coefficients sont continus et bornés dans la région R ; plus précisément, supposons l'existence des nombres positifs A, α et γ tels que: $|A_{ik}| < A, |a_j| < \alpha, |c| < \gamma, (i, k, j = 1, 2, \dots, m)$, dans toute la région R ; quant au coefficient b , on suppose l'existence des nombres β et β_1 tels que $\beta_1 > b > \beta > 0$ partout dans R . Tout ceci étant supposé, nous allons démontrer le théorème suivant:

THÉORÈME I. *La seule solution de l'équation (1) appartenant à la classe E_2 par rapport aux x_i et s'annulant sur la caractéristique $y=0$ est $u \equiv 0$.*

DÉMONSTRATION. D'après la règle générale due à M. Picone (voir le travail cité) on transforme l'équation (1) en posant $u(x_1, x_2, \dots, x_m, y) = v(x_1, x_2, \dots, x_m, y) \cdot W(x_1, x_2, \dots, x_m, y)$, v étant la nouvelle fonction inconnue; W étant supposé différent de zéro, on obtient ainsi

$$(2) \quad \sum_{i,k=1}^m A_{i,k} \frac{\partial^2 v}{\partial x_i \partial x_k} = \sum_{j=1}^m a_j^{(1)} \frac{\partial v}{\partial x_j} + b \frac{\partial v}{\partial y} + c^{(1)} v$$

et on doit choisir W de sorte que 1° le coefficient $c^{(1)}$ soit positif dans R , 2° $v \rightarrow 0$ pour $\sum_{i=1}^m x_i^2 \rightarrow \infty$.

Passons donc à la construction de la fonction W conforme à notre problème. La fonction u appartenant à la classe E_2 , il existe deux nombres M et K_0 tels que

$$(3) \quad |u(x_1, x_2, \dots, x_m, y)| < M \exp \left(K_0 \cdot \sum_{i=1}^m x_i^2 \right)$$

dans R . Soit

$$\mu = \frac{5K}{\beta}; \quad \nu = \frac{1}{\beta} \left[m \left(1 + \frac{\alpha}{A} \right)^2 + K + \gamma \right]$$

où $K > K_0$ et posons

$$u(x_1, x_2, \dots, x_m, y) = v(x_1, x_2, \dots, x_m, y) \cdot H(x_1, x_2, \dots, x_m, y; K),$$

avec

$$(4) \quad H(x_1, x_2, \dots, x_m, y; K) = \exp \left[\frac{K \sum_{i=1}^m x_i^2}{1 - \mu y} + \nu y \right].$$

Supposons que la hauteur h de R est inférieure à $1/\mu$.

Ainsi dans la transformation de M. Picone nous avons remplacé la fonction W par H . Un calcul montre qu'en chaque point de R le coefficient $c^{(1)}$ de l'équation (2) est supérieur à une forme quadratique en x_i positive définie, à savoir:

$$c^{(1)} > A \sum_{j=1}^m \left[\frac{K |x_j|}{1 - \mu y} - \left(1 + \frac{\alpha}{A} \right) \right]^2 + K.$$

Il en résulte que si l'on construit un parallélépipède rectangulaire R_n , détaché de R par les plans $x_i = \pm n$, la fonction v ne pourrait atteindre dans R_n ni un maximum positif, ni un minimum négatif que sur la partie de sa surface située sur les plans $x_i = \pm n$ et $y = 0$ (voir le travail cité de M. Picone).

Soit $P_0(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)})$ un point quelconque de R ; pour n assez grand R_n contiendra P_0 . En vertu de (3) on peut déterminer n de sorte que $|v| < \epsilon$ sur la surface latérale de R_n , $\epsilon > 0$ étant un nombre arbitrairement petit. D'autre part v s'annule sur le plan $y = 0$, donc $|v(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)})| < \epsilon$. Il en résulte que $v(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)}) = 0$ et, par suite, $u(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)}) = 0$. Comme P_0 est un point arbitraire de R , on a

$$u(x_1, x_2, \dots, x_m, y) \equiv 0$$

dans R .

Passons à la démonstration de l'existence de la solution du problème, en supposant que les coefficients $A_{i,k}$ et b admettent les dérivées du premier ordre continues.

THÉORÈME II. La fonction $\phi(x_1, x_2, \dots, x_m)$ étant de la classe E_2 et continue, il existe une solution et une seule de l'équation (1) appartenant à la classe E_2 en tant que la fonction des x_i déterminée dans la région R de hauteur h (suffisamment petite) et se réduisant à la fonction ϕ sur la caractéristique $y = 0$.

DÉMONSTRATION. L'unicité de cette solution résulte aussitôt du Théorème I. Il suffit donc de démontrer son existence.

La fonction ϕ étant de la classe E_2 , il existe deux nombres M et K tels que

$$(5) \quad |\phi(x_1, x_2, \dots, x_m)| < M \exp \left(K \sum_{i=1}^m x_i^2 \right).$$

Construisons la parallélépipède R_p , analogue à R_n (voir la démonstration du Théorème I) et soit S_p sa surface latérale, composé des parties des plans $x_i = \pm p$. D'après les résultats de M. Giraud et

M. Gevrey³ il existe une solution u_p de (1) telle que:

$$\begin{aligned} u_p(x_1, x_2, \dots, x_m, 0) &= \phi(x_1, x_2, \dots, x_m), \\ u_p(x_1, x_2, \dots, x_m, y) &= \phi(x_1, x_2, \dots, x_m) \quad \text{sur } S_p. \end{aligned}$$

Cette solution est déterminée dans R_p . D'une manière analogue définissons la solution u_q de (1), déterminée dans le parallépipède R_q ($q > p$) analogue à R_p ; soit S_q sa surface latérale.

Supposons que $h < 2/KA$ et soient v_p, v_q et v_q^* les fonctions définies par les égalités

$$\begin{aligned} u_p &= v_p \cdot H(x_1, x_2, \dots, x_m, y; 2K); \quad u_q = v_q \cdot H(x_1, x_2, \dots, x_m, y; 2K); \\ u_q &= v_q^* \cdot H(x_1, x_2, \dots, x_m, y; K). \end{aligned}$$

Observons que $H(x_1, x_2, \dots, x_m, y; 2K) > H(x_1, x_2, \dots, x_m, y; K) \cdot \exp(\bar{K} \sum_{i=1}^m x_i^2)$ ($\bar{K}$ étant un nombre positif) de sorte que

$$(6) \quad v_q < v_q^* \exp\left(-\bar{K} \sum_{i=1}^m x_i^2\right);$$

on peut supposer que $\bar{K} < K$. D'après (5) et la définition de H on a $|v_q^*| < M$ sur S_q et $|v_q^*(x_1, x_2, \dots, x_m, 0)| < M$; il en résulte que $|v_q^*| < M$ partout dans R_q et en particulier sur S_p . En égard à (6) on a donc $|v_q| < M e^{-\bar{K} m p^2}$ sur S_p .

La fonction:

$$w_{pq}(x_1, x_2, \dots, x_m, y) = v_q(x_1, x_2, \dots, x_m, y) - v_p(x_1, x_2, \dots, x_m, y)$$

est déterminée partout dans R_p et l'on a $|w_{pq}| < 2M \exp(-\bar{K} m p^2)$ sur S_p et $w_{pq}(x_1, x_2, \dots, x_m, 0) = 0$. Donc

$$(7) \quad |w_{pq}(x_1, x_2, \dots, x_m, y)| < 2M e^{-\bar{K} m p^2} \quad \text{partout dans } R_p.$$

Soit maintenant P_0 un point quelconque de R ; considérons un parallépipède fixe $\rho \subset R$ de hauteur h , entourant P_0 . D'après (7) on peut déterminer un nombre p_0 de sorte que l'on ait

$$|u_q(x_1, x_2, \dots, x_m, y) - u_p(x_1, x_2, \dots, x_m, y)| < \epsilon$$

pour $q > p > p_0$ partout dans ρ . Ceci montre que la suite $\{u_p\}$ tend vers une limite $U(x_1, x_2, \dots, x_m, y)$ partout dans R , le point P_0 étant un point arbitraire de R ; en particulier $U_p \rightarrow U$ uniformément dans ρ .

Nous démontrons que U_0 est précisément la solution de notre

³ G. Giraud, *Sur certaines opérations aux dérivées partielles du type parabolique*, Comptes Rendus de l'Académie des Sciences, Paris, Tome 195 (1932), pp. 98-100. M. Gevrey, *Systèmes d'équations aux dérivées partielles du type parabolique*, ibid., Tome 195 (1932), pp. 690-693.

problème. On a évidemment $U(x_1, x_2, \dots, x_m, y) = \phi(x_1, x_2, \dots, x_m)$; il nous reste à prouver que U satisfait à l'équation (1) partout dans R . Il suffit évidemment de démontrer que ceci a lieu au point P . Or soit $\bar{u}(x_1, x_2, \dots, x_m, y)$ la solution de (1) déterminée dans ρ , égale à ϕ pour $y=0$ et identique à U sur la surface latérale σ de ρ . Nous allons voir que $\bar{u} \equiv U$ dans ρ .

En effet, $\epsilon > 0$ étant un nombre arbitrairement petit, on peut déterminer le nombre p de sorte que

$$\begin{aligned} |u_p(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)}) - U(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)})| &< \epsilon/2, \\ |u_p(x_1, x_2, \dots, x_m, y) - U(x_1, x_2, \dots, x_m, y)| &< \epsilon/2 \quad \text{sur } \sigma. \end{aligned}$$

Il résulte de la dernière inégalité, que

$$|u_p(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)}) - \bar{u}(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)})| < \epsilon/2$$

puisque $u_p - \bar{u}$ est dans ρ une solution de (1) et on a $u_p = \bar{u}$ pour $y=0$ et $\bar{u} \equiv U$ sur σ . On en déduit $|\bar{u}(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)}) - U(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)})| < \epsilon$ donc

$$\bar{u}(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)}) = U(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)}, y^{(0)}).$$

Ainsi U est une intégrale de (1) dans R , c'est donc la solution de notre problème.

Il nous reste encore à démontrer qu'elle est de la classe E_2 par rapport aux x_i . Or la fonction $v^* = \lim_{p \rightarrow \infty} v_p^*$ satisfait à l'inégalité $|v^*| < M$. Comme d'autre part $U = v^* \cdot H(x_1, x_2, \dots, x_m, y; K)$, on a

$$|U(x_1, x_2, \dots, x_m, y)| < M \exp \left[\frac{K \sum_{i=1}^m x_i^2}{1 - \mu y} + \nu y \right].$$

Si la fonction ϕ est de la classe E_1 , on démontrerait aisément, qu'il en est de même de U (par rapport aux x_i); plus précisément, l'inégalité $|\phi| < M \exp [K(\sum_{i=1}^m x_i^2)^{1/2}]$ entraîne

$$|U| < M \exp K \left[\left(\sum_{i=1}^m x_i^2 \right)^{1/2} + \Gamma(y) \right],$$

Γ étant une fonction bornée. En particulier l'inégalité $|\phi| < M$ entraîne $|U| < M$ dans R .

ON APPROXIMATION BY EUCLIDEAN AND NON-EUCLIDEAN TRANSLATIONS OF AN ANALYTIC FUNCTION

W. SEIDEL AND J. L. WALSH

In 1929 G. D. Birkhoff established¹ the noteworthy result that an entire function $F(z)$ exists such that to an arbitrary entire function $g(z)$ corresponds a sequence $a_1, a_2, \dots$ depending on $g(z)$ with the property

$$(1) \quad \lim_{n \rightarrow \infty} F(z + a_n) = g(z)$$

for all z , uniformly for z on every closed bounded set.

It is the object of the present note (a) to indicate that not merely an arbitrary entire function $g(z)$ can be expressed in the form (1), but also any function analytic in a simply connected region, and (b) to study the non-euclidean analogue of the entire problem; precisely analogous results are obtained. Some related topics under (a) have recently been studied by A. Roth,² who, however, does not mention the results to be proved here.

The immediate occasion of the interest of the present writers³ in the problem is through (b), for non-euclidean translations have been widely used in the study of derivatives of univalent and other functions analytic in the unit circle $|z|=1$; limit functions under such translations are of great significance in the study of derivatives and of limit values of a given function as a variable point z approaches the circumference $|z|=1$.

We shall give a proof of the following theorem, proof and theorem differing only in detail from those of Birkhoff:

THEOREM 1. *There exists an entire function $F(z)$ such that given an arbitrary function $f(z)$ analytic in a simply connected region R of the z -plane, we have for suitably chosen $a_1, a_2, \dots$ the relation*

$$(2) \quad \lim_{n \rightarrow \infty} F(z + a_n) = f(z)$$

for z in R , uniformly on any closed bounded set in R .

¹ Comptes Rendus de l'Académie des Sciences, Paris, vol. 189, pp. 473-475.

² Commentarii Mathematici Helvetici, vol. 11 (1938-1939), pp. 77-125.

³ Compare Seidel and Walsh, *On the derivatives of functions analytic in the unit circle and their radii of univalence and of p -valence*, a forthcoming paper in the Transactions of this Society.

Here and throughout the sequel we are concerned with the finite plane, that is to say, the plane of finite points z .

By way of geometric entities, we introduce the circles $C_1: |z-4|=2$, $C_2: |z-4^2|=2^2, \dots, C_n: |z-4^n|=2^n, \dots$, and also the circles $\Gamma_n: |z|=4^n+2^n+1$; it follows that the C_n are mutually exterior, and that Γ_n contains in its interior all the circles $C_1, C_2, \dots, C_n$ but no point in or on any of the circles $C_{n+1}, C_{n+2}, \dots$.

Let us enumerate the polynomials in z with rational coefficients: $p_1(z), p_2(z), \dots$. It is of course true that any sequence of polynomials can be replaced by a sequence of polynomials with rational coefficients, without altering whatever properties may exist of convergence or uniform convergence to a given function on bounded point sets.

We choose $\pi_1(z)$ as a polynomial in z which satisfies the inequality $|p_1(z-4) - \pi_1(z)| < 1/2$, z on or within C_1 ; indeed we may choose $\pi_1(z) \equiv p_1(z-4)$. We choose $\pi_2(z)$ as a polynomial in z which satisfies the two inequalities

$$\begin{aligned} |\pi_1(z) - \pi_2(z)| &< 1/4, & z \text{ on or within } \Gamma_1, \\ |p_2(z-4^2) - \pi_2(z)| &< 1/4, & z \text{ on or within } C_2; \end{aligned}$$

such a polynomial $\pi_2(z)$ exists, by Runge's classical theorem. In general, let $\pi_n(z)$ be a polynomial in z which satisfies the inequalities

$$\begin{aligned} |\pi_{n-1}(z) - \pi_n(z)| &< 1/2^n, & z \text{ on or within } \Gamma_{n-1}, \\ |p_n(z-4^n) - \pi_n(z)| &< 1/2^n, & z \text{ on or within } C_n. \end{aligned}$$

The sequence $\{\pi_n(z)\}$ converges uniformly in each of the circles Γ_n ; hence converges at every point of the plane, uniformly on any bounded set. The limit function $F(z)$ is entire, and has the required properties. Indeed, let $f(z)$ be analytic in a simply connected region R ; there exist polynomials $p_{n_h}(z)$ of the set already defined with

$$(3) \quad \lim_{n_h \rightarrow \infty} p_{n_h}(z) = f(z)$$

at every point of R , uniformly on any closed bounded set in R . For z in $C_n: |z-4^n| < 2^n$ we have

$$\begin{aligned} F(z) &= \pi_n(z) + [\pi_{n+1}(z) - \pi_n(z)] \\ &\quad + [\pi_{n+2}(z) - \pi_{n+1}(z)] + \dots, \\ |F(z) - p_n(z-4^n)| &\leq |p_n(z-4^n) - \pi_n(z)| + |\pi_{n+1}(z) - \pi_n(z)| \\ &\quad + |\pi_{n+2}(z) - \pi_{n+1}(z)| + \dots \\ &< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots = \frac{1}{2^{n-1}}, \end{aligned}$$

whence

$$(4) \quad \lim_{n \rightarrow \infty} [F(z + 4^n) - p_n(z)] = 0$$

for every z , uniformly on any bounded set. To return to $f(z)$, we now have from (3) and (4)

$$(5) \quad \lim_{n_k \rightarrow \infty} [F(z + 4^{n_k}) - f(z)] = 0$$

for z in R , uniformly on any closed bounded set in R . Theorem 1 is established.

The special case of Theorem 1 that $f(z)$ is an entire function and R is the (finite) z -plane is included here, and is the case considered by Birkhoff. We add the remark that whenever a function $g(z)$ can be represented on a point set E (bounded or unbounded) by a sequence of polynomials, that function can also be represented on E in the manner indicated by (2), with preservation of the property of uniform convergence whenever that occurs on a bounded set belonging to E . For instance, E may consist of a sequence of disjoint simply connected regions $R_1, R_2, \dots$, with $g(z)$ analytic on E ; then $g(z)$ can be represented on E either by a sequence of polynomials or, as in (2), with uniform convergence on any closed bounded subset of E . On the general subject of representation by polynomials there exist modern researches due to Montel, Walsh, Hartogs and Rosenthal, and Lavrentieff.⁴

A further remark in connection with Theorem 1 is that *if the numbers $A_0, A_1, A_2, \dots$ are arbitrary, there exists a sequence $a_1, a_2, \dots$ with the property*

$$(6) \quad \lim_{n \rightarrow \infty} F^{(k)}(a_n) = A_k, \quad k = 0, 1, 2, \dots$$

To establish (6) it is sufficient to remark that when m is given, the number a_m exists with the property

$$\left| F(z + a_m) - \left[A_0 + A_1 z + \frac{A_2}{2!} z^2 + \dots + \frac{A_m}{m!} z^m \right] \right| < \frac{1}{2^m \cdot m!},$$

for $|z| \leq 1$;

from Cauchy's inequality it then follows that we have $|F^{(k)}(a_m) - A_k| < 1/2^m$, $k = 0, 1, 2, \dots, m$; the relation (6) follows.

⁴ The reader may refer to Lavrentieff, *Sur les Fonctions d'une Variable Complexe Représentables par des Séries de Polynômes*, Actualités Scientifiques et Industrielles, no. 441, Paris, 1936.

We turn now to the non-euclidean analogue of Theorem 1:

THEOREM 2. *There exists a function $\Phi(z)$ analytic in the region $|z| < 1$ such that given an arbitrary function $\phi(z)$ analytic in a simply connected subregion R , we have for suitably chosen $\alpha_1, \alpha_2, \dots$ the relation*

$$(7) \quad \lim_{n \rightarrow \infty} \Phi \left(\frac{z + \alpha_n}{1 + \bar{\alpha}_n z} \right) = \phi(z),$$

for z in R , uniformly on any closed set interior to R .

As in Theorem 1 we needed to use only real a_n , so here we shall actually employ only real α_n .

For geometric entities we choose here C_1 as the n.e. circle of n.e. radius 2 whose n.e. center is the point $z = \beta_1$ of the axis of reals whose n.e. distance from $z = 0$ is 4, and in general choose C_n as the n.e. circle of n.e. radius 2^n whose n.e. center is the point $z = \beta_n$ of the axis of reals whose n.e. distance from $z = 0$ is 4^n . Let Γ_n be the circle whose center is $z = 0$ and n.e. radius $4^n + 2^n + 1$, so that Γ_n contains in its interior all the circles $C_1, C_2, \dots, C_n$, but no point in or on any of the circles $C_{n+1}, C_{n+2}, \dots$.

As before, we use the polynomials $p_1(z), p_2(z), \dots$ with rational coefficients. Choose $\pi_1(z)$ as a polynomial in z which satisfies the inequality

$$\left| p_1 \left(\frac{z - \beta_1}{1 - \beta_1 z} \right) - \pi_1(z) \right| < 1/2, \quad z \text{ on or within } C_1;$$

choose $\pi_2(z)$ as a polynomial in z which satisfies the two inequalities

$$\begin{aligned} |\pi_1(z) - \pi_2(z)| &< 1/4, \quad z \text{ on or within } \Gamma_1, \\ \left| p_2 \left(\frac{z - \beta_2}{1 - \beta_2 z} \right) - \pi_2(z) \right| &< 1/4, \quad z \text{ on or within } C_2. \end{aligned}$$

In general, let $\pi_n(z)$ be a polynomial in z which satisfies

$$\begin{aligned} |\pi_{n-1}(z) - \pi_n(z)| &< 1/2^n, \quad z \text{ on or within } \Gamma_{n-1}; \\ \left| p_n \left(\frac{z - \beta_n}{1 - \beta_n z} \right) - \pi_n(z) \right| &< 1/2^n, \quad z \text{ on or within } C_n. \end{aligned}$$

The sequence $\{\pi_n(z)\}$ converges uniformly in each of the circles Γ_n , hence converges at every point of the region $|z| < 1$, uniformly on any closed subset. The limit function $\Phi(z)$ is analytic throughout the region $|z| < 1$, and will now be shown to have the required properties.

For z in C_n we have

$$\begin{aligned}\Phi(z) &= \pi_n(z) + [\pi_{n+1}(z) - \pi_n(z)] \\ &\quad + [\pi_{n+2}(z) - \pi_{n+1}(z)] + \cdots, \\ \left| \Phi(z) - p_n\left(\frac{z - \beta_n}{1 - \beta_n z}\right) \right| &\leq \left| p_n\left(\frac{z - \beta_n}{1 - \beta_n z}\right) - \pi_n(z) \right| \\ &\quad + |\pi_{n+1}(z) - \pi_n(z)| + \cdots \\ &< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots = \frac{1}{2^{n-1}},\end{aligned}$$

whence

$$(8) \quad \lim_{n \rightarrow \infty} \left[\Phi\left(\frac{z + \beta_n}{1 + \beta_n z}\right) - p_n(z) \right] = 0$$

for every z in $|z| < 1$, uniformly on any closed set in $|z| < 1$.

Let $\phi(z)$ be analytic in the simply connected region R in $|z| < 1$. There exist polynomials $p_{n_k}(z)$ of the set already defined with

$$(9) \quad \lim_{n_k \rightarrow \infty} p_{n_k}(z) = \phi(z)$$

at every point of R , uniformly on any closed subset. From (8) and (9) we find

$$(10) \quad \lim_{n_k \rightarrow \infty} \left[\Phi\left(\frac{z + \beta_{n_k}}{1 + \beta_{n_k} z}\right) - \phi(z) \right] = 0$$

for z in R , uniformly on any closed set in R .

Theorem 2 is established. The region R may in particular be the region $|z| < 1$. Remarks for Theorem 2 entirely analogous to those for Theorem 1 are also valid.

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ZERO-DIMENSIONAL FAMILIES OF SETS¹

SAMUEL EILENBERG AND E. W. MILLER

A family $\Phi = \{A_\alpha\}$ of subsets of a topological space X will be called *0-dimensional* if given an open set U such that $A_{\alpha_0} \subset U$, there is an open set V such that (1) $A_{\alpha_0} \subset V \subset U$ and (2) $(\bar{V} - V) \sum_\alpha A_\alpha = 0$. We enumerate below a few of the most common 0-dimensional families. In each case the proof of 0-dimensionality is easy, and is therefore omitted.

(I) Every family of disjoint open subsets of a topological space is 0-dimensional.

(II) Let Y be a locally connected subset of a topological space X . The family Φ of the components of Y is 0-dimensional.

(III) Let Y be a compact and closed subset of a metric space X . The family Φ of the components of Y is 0-dimensional.

(IV) Let Y be a subset of a metric space X . The family Φ consisting of the individual points of Y is 0-dimensional if and only if $\dim Y = 0$.

(V) Let Φ be a family of closed subsets of a compact metric space X . If, given any sequence $F, F_1, F_2, \dots$ of sets of Φ , the relation $F \cdot \liminf F_i \neq 0$ implies $\liminf F_i \subset F$, then the family Φ is called *upper-semi-continuous*. In this case the sets of the family Φ are disjoint. There is a standard way of introducing a topology into the family Φ which leads to a separable metrizable *hyperspace* Φ^* . The family Φ is 0-dimensional if and only if $\dim \Phi^* = 0$. In particular, Φ is 0-dimensional whenever it is upper-semi-continuous and countable.

(VI) Let Y be a subset of a topological space X and let Y be homeomorphic with a subset of the linear continuum. The family Φ of the components of Y is 0-dimensional.

The purpose of this note is to establish the following theorem:

THEOREM. *Let X be a unicoherent Peano continuum,² $\Phi = \{A_\alpha\}$ a 0-dimensional family of subsets of X , and x_1 and x_2 two points of X . If none of the sets A_α cuts X between x_1 and x_2 ,³ then $\sum_\alpha A_\alpha$ does not cut X between x_1 and x_2 .*

Various corollaries can be obtained by taking X to be the n -sphere

¹ Presented to the Society, December 26, 1939, under the title *On 0-dimensional upper-semi-continuous collections*.

² A Peano continuum X is called *unicoherent* if given any decomposition $X = X_1 + X_2$ into continua, the set $X_1 \cdot X_2$ is a continuum.

³ A set $A \subset X$ cuts X between x_1 and x_2 if $X - A$ contains no continuum joining x_1 and x_2 .

S^n ($n > 1$) or the n -cube Q^n ($n > 0$) and Φ to be one of the families (I)–(VI).⁴

We shall establish two lemmas before giving the proof of the theorem.

LEMMA 1. *Let X be a unicoherent Peano continuum. Let A_1 and A_2 be open and disjoint subsets of X , and let A_2 be connected. If neither A_1 nor A_2 cuts X between the points x_1 and x_2 , then $A_1 + A_2$ does not cut X between x_1 and x_2 .*

PROOF. Let C_2 be the component of $X - A_2$ which contains x_1 and x_2 . Let $B_1 = A_1 \cdot C_2$ and $B_2 = X - C_2$. It follows easily that B_1 and B_2 are open, $B_1 \cdot B_2 = 0$, B_2 and $X - B_2 = C_2$ are connected, and $A_1 + A_2 \subset B_1 + B_2$. It will be sufficient to show that $B_1 + B_2$ does not cut X between x_1 and x_2 .

Since $B_1 \subset A_1$, the set B_1 does not cut X between x_1 and x_2 . We will denote the component of $X - B_1$ which contains x_1 and x_2 by C_1 . If $C_1 \cdot B_2 = 0$, then $C_1 \subset X - (B_1 + B_2)$, so that $B_1 + B_2$ does not cut X between x_1 and x_2 . We will suppose then that $C_1 \cdot B_2 \neq 0$. Since C_1 and B_2 are connected it follows that $C_1 + B_2$ is connected. But $C_1 + B_2 \subset X - B_1$. Hence $C_1 + B_2 \subset C_1$, and therefore $B_2 \subset C_1$. Since $C_2 = X - B_2$, it follows that $X = C_1 + C_2$. Since C_1 and C_2 are continua and X is unicoherent, it follows that $C_1 \cdot C_2$ is a continuum. But x_1 and x_2 belong to $C_1 \cdot C_2$, and $C_1 \cdot C_2 \subset X - (B_1 + B_2)$. Hence $B_1 + B_2$ does not cut X between x_1 and x_2 .

LEMMA 2. *Let X be a unicoherent Peano continuum and let $A_1, A_2, \dots, A_n, \dots$ be a sequence of disjoint open subsets of X . If none of the sets A_n cuts X between x_1 and x_2 , then $\sum_{n=1}^{\infty} A_n$ does not cut X between x_1 and x_2 .*

PROOF. Since X is locally connected and separable, every open set in X consists of a countable number of components each of which is open. It is clear, then, that we may assume each set A_n to be connected.

Let k be any positive integer. Using Lemma 1, it follows by finite induction that $A_1 + A_2 + \dots + A_k$ does not cut X between x_1 and x_2 . Let C_k be the component of $X - (A_1 + A_2 + \dots + A_k)$ which contains x_1 and x_2 . Then $\prod_{k=1}^{\infty} C_k$ is a continuum containing x_1 and x_2 , and $\prod_{k=1}^{\infty} C_k \subset X - \sum_{k=1}^{\infty} A_k$. Hence $\sum_{k=1}^{\infty} A_k$ does not cut X between x_1 and x_2 .

⁴ See R. L. Moore, Proceedings of the National Academy of Sciences, vol. 10 (1934), p. 356, and S. Eilenberg, Fundamenta Mathematicae, vol. 26 (1936), pp. 76–77.

We now return to the *proof of the theorem*:

For any F in Φ there is an open set $U(F)$ such that $F \subset U(F)$, and $U(F)$ does not cut X between x_1 and x_2 . Since Φ is 0-dimensional, there is an open set $V(F)$ such that $F \subset V(F) \subset U(F)$ and $[\overline{V(F)} - V(F)] \sum F = 0$. By the Lindelöf covering theorem, there is a sequence $F_1, F_2, \dots$ of elements of Φ such that $\sum F \subset \sum_{i=1}^{\infty} V(F_i)$. Now let

$$\begin{aligned} A_1 &= V(F_1), & A_2 &= V(F_2) - \overline{V(F_1)}, \\ A_k &= V(F_k) - \overline{[V(F_1) + \dots + V(F_{k-1})]}, \dots \end{aligned}$$

The sets $A_1, A_2, \dots, A_k, \dots$ are open and disjoint, and no one of them cuts X between x_1 and x_2 . But, as is easily shown, $\sum F \subset \sum_{k=1}^{\infty} A_k$. Hence in view of Lemma 2, $\sum F$ does not cut X between x_1 and x_2 .

THE UNIVERSITY OF MICHIGAN

SUMS OF FOURTH POWERS OF GAUSSIAN INTEGERS

IVAN NIVEN

It is the purpose of this note to give necessary and sufficient conditions for the expressibility of a Gaussian integer as a sum of fourth powers of Gaussian integers; and then to determine an upper bound to the number of fourth powers necessary when the conditions are satisfied. Our results are as follows:

THEOREM. *A Gaussian integer is expressible as a sum of fourth powers of Gaussian integers if and only if its imaginary coordinate is divisible by 24. Every integer $a + 24bi$, where a and b are rational integers, is expressible as a sum of 18 or fewer fourth powers.*

First we prove that the condition is necessary. We note that¹

$$(1) \quad (x + yi)^4 = x^4 - 6x^2y^2 + y^4 + 4ixy(x^2 - y^2).$$

It is obvious that $xy(x^2 - y^2)$ is divisible by 2 and by 3. Hence any fourth power has an imaginary coordinate divisible by 24, and any sum of fourth powers has the same property.

The converse of this is included in the second statement in the theorem, which we now proceed to prove. The author² has shown that a Gaussian integer $a + 2bi$ is expressible as a sum of two squares

¹ Latin letters will represent rational integers throughout this paper.

² *Integers of quadratic fields as sums of squares*, Transactions of this Society, vol. 48 (1940), p. 410, Theorem 2.

if and only if not both $a/2$ and b are odd rational integers. We use this result in the following two lemmas.

LEMMA 1. *The Gaussian integer $6(c+di)^2$, where c and d are odd and even, respectively, is expressible as a sum of six fourth powers.*

By the theorem stated above, we can write $c+di=\alpha^2+\beta^2$, where α and β are Gaussian integers. Also we employ the identity

$$6(\alpha^2 + \beta^2)^2 = 2(\alpha + \beta)^4 + 2(\alpha - \beta)^4 + (\alpha + \beta i)^4 + (\alpha - \beta i)^4,$$

which completes the proof.

LEMMA 2. *A Gaussian integer of the form $48h+6k+24mi$, where k equals 2 or 6 according as m is even or odd, is expressible as a sum of twelve fourth powers.*

As above, we can write

$$(2) \quad 8h + k + 4mi = (c + di)^2 + (e + fi)^2,$$

which implies

$$(3) \quad 8h + k = c^2 + e^2 - d^2 - f^2, \quad 2m = cd + ef.$$

Since $8h+k \equiv 2 \pmod{4}$, the first of equations (3) shows that either c and e are odd and d and f are even, or vice versa. We show that the latter cannot be the case.

First let m be even, so that $k=2$. Then equations (3) imply the congruences

$$(4) \quad c^2 + e^2 - d^2 - f^2 \equiv 2 \pmod{8}, \quad cd + ef \equiv 0 \pmod{4}.$$

Since the square of an odd integer is congruent to 1 modulo 8, the assumption that d and f are odd leads to the congruence $c^2 + e^2 \equiv 4 \pmod{8}$. Hence the integers c and e are even, but are incongruent modulo 4. But these conditions on c and e are incompatible with the second of the congruences (4).

On the other hand, if m is odd, $k=6$, and corresponding to (4) we have the congruences

$$(5) \quad c^2 + e^2 - d^2 - f^2 \equiv 6 \pmod{8}, \quad cd + ef \equiv 2 \pmod{4}.$$

Assuming again that d and f are odd, we obtain $c^2 + e^2 \equiv 0 \pmod{8}$. Hence c and e are even, and, moreover, are congruent modulo 4. These conditions are such that the second of congruences (5) has no solutions.

Thus we have proven that d and f are even, and c and e are odd.

Multiplying equation (2) by 6, we see that Lemma 1 is applicable to each term on the right side, and we have the desired result.

To complete the proof of the theorem, we prove this lemma:

LEMMA 3. *Any integer $a+24bi$ is expressible as a sum of six fourth powers and an integer of the type described in Lemma 2.*

We identify b with m , and show that a is congruent to six real fourth powers modulo $48h+6k$. This we do by exhibiting the numbers $0, 1, \dots, 47$ as sums of fourth powers modulo 48. Most of these can be handled by the use of: $1^4=1$, $2^4=16$, $(1+i)^4=-4$. Except for the values 7, 22, 23, 27, 37, 38, 39, 42, 43, and 47, all integers from 1 to 47 can be expressed as sums of 1, 16, and -4 , not more than six summands being used in each case. For example, we have

$$11 = 16 - 4 - 4 + 1 + 1 + 1, \quad 46 = 16 + 16 + 16 - 4 + 1 + 1.$$

Turning now to the exceptional cases, we make use of $81=3^4$, and introduce congruences modulo 48. We can write

$$\begin{aligned} 7 &\equiv 151 = 81 + 81 - 4 - 4 - 4 + 1, \\ 22 &\equiv 70 = 81 - 4 - 4 - 4 + 1, \\ 27 &\equiv 75 = 81 - 4 - 4 + 1 + 1, \\ 37 &\equiv -11 = -4 - 4 - 4 + 1, \\ 42 &\equiv -6 = -4 - 4 + 1 + 1, \\ 47 &\equiv -1 = -4 + 1 + 1 + 1, \end{aligned}$$

these being congruences modulo 48. Since the integer 37 is represented here as a sum of four fourth powers modulo 48, the integers 38 and 39 are similarly sums of five and six fourth powers, respectively. Also the integers 23 and 43 can be compared with 22 and 42 above.

Although the theorem has been proved completely, we now show that Lemma 3 cannot be improved, that is, that six fourth powers are necessary in at least one case. Consider the situation when $a=19+48A$, $b=2B$. In this case we can show that it is not possible to obtain rational integers h and m , and Gaussian integers $\alpha_1, \dots, \alpha_5$ to satisfy

$$(6) \quad a + 24bi = \sum_{i=1}^5 \alpha_i^4 + 48h + 6k + 24mi,$$

where, of course, k is 2 or 6 according as m is even or odd.

LEMMA 4. *If the imaginary part of the fourth power of a Gaussian integer is congruent to 0 modulo 48, the real part is congruent to 0, 1, or 12*

modulo 16. If the imaginary part is congruent to 24 modulo 48, the real part is congruent to 9 modulo 16.

From equation (1) it follows that

$$R(x + yi)^4 = (x^2 - y^2)^2 - 4x^2y^2, \quad I(x + yi)^4 = 4xy(x^2 - y^2).$$

Since the square of a rational integer is congruent to 0, 1, 4 or 9 modulo 16, it is easily verified that $R(x + yi)^4$ is congruent to 0, 1, 9, or 12 modulo 16. Also note that $R(x + yi)^4$ is congruent to 9 modulo 16 if and only if x is odd and y is congruent to 2 modulo 4, or vice versa; and when x and y satisfy these conditions, $I(x + yi)^4$ is congruent to 24 modulo 48. In all other cases either xy or $x^2 - y^2$ is divisible by 4, and hence $I(x + yi)^4$ is congruent to 0 modulo 48.

We now use Lemma 4 to prove equation (6) impossible. First let m be even, so that $k=2$. Since b is even the imaginary parts of (6) can be equal only if $\sum \alpha_i^4$ has an imaginary coordinate congruent to 0 modulo 48. This implies that an even number of the terms $I(\alpha_i^4)$ are congruent to 24 modulo 48. In turn, Lemma 4 states that an even number of the terms $R(\alpha_i^4)$ are congruent to 9 modulo 16. The real parts of equation (6) can be written as a congruence:

$$19 \equiv \sum_{i=1}^5 R(\alpha_i^4) + 12 \pmod{16}.$$

Simple verification shows that this congruence cannot be satisfied by assigning the values 0, 1, 9, or 12 to the terms $R(\alpha_i^4)$, the value 9 being used an even number of times.

Second let m be odd, so that $k=6$. Corresponding to the above congruence we have

$$19 \equiv \sum_{i=1}^5 R(\alpha_i^4) + 36 \pmod{16}.$$

In this case, however, Lemma 4 requires that we have an odd number of the terms $R(\alpha_i^4)$ congruent to 9 modulo 16. Again it can be verified that the congruence has no solutions satisfying this condition.

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THE GEOMETRY OF WHIRLS AND WHIRL-MOTIONS IN SPACE¹

J. M. FELD

1. **Introduction.** The geometry of *whirls* and *whirl-motions* in the plane was inaugurated by E. Kasner.² An adaptation of Kasner's geometry to the sphere was made by K. Strubecker.³ It is the purpose of this paper to develop a strictly analogous geometry in euclidean three-space, S_3 . To render such a development possible we shall introduce a new type of *oriented* plane element—namely, a geometric object formed by a plane, a point in the plane, and an ordered pair of orthogonal fundamental directions in the plane; the fundamental directions shall be given by a pair of unit vectors in the plane. There are $2\infty^6$ such plane elements in S_3 ; there are ∞^5 of the Lie kind. Henceforth, *plane element* shall mean only the new kind of plane element.⁴

2. **Turns, slides, and direct whirls.** Let e_0, e_1, e_2, e_3 be the Hamiltonian quaternion units such that

$$e_0 e_i = e_i e_0 = e_i, \quad e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1.$$

Let any real point P in S_3 whose orthogonal cartesian coordinates are z_1, z_2, z_3 be represented by the position vector $z = z_1 e_1 + z_2 e_2 + z_3 e_3$. Let

¹ Presented to the Society February, 22, 1941.

² Edward Kasner, *The group of turns and slides and the geometry of turbines*, American Journal of Mathematics, vol. 33 (1911), pp. 193–202. Further development of the subject appears in a series of papers by Kasner and De Cicco; see, for example, their joint papers, *Quadratic fields in the geometry of the whirl-motion group G_6* , *ibid.*, vol. 61 (1939), pp. 131–142; and *The geometry of the whirl-motion group G_6 : elementary invariants*, this Bulletin, vol. 44 (1938), pp. 399–403.

³ K. Strubecker, *Zur Geometrie sphärischer Kurvenscharen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 44 (1934), pp. 184–198.

⁴ Inasmuch as our *oriented plane element* defines a position of a rigid body in space, it is essentially equivalent to Study's *soma*, *Geometrie der Dynamen*, Leipzig, 1903, and to De Saussure's *feuillet*, *Exposé Résumé de la Géométrie des Feuilles*, Geneva, 1910; see also R. Bricard, *Nouvelles Annales de Mathématiques*, (4), vol. 10 (1910). It was remarked by Kasner in his 1911 paper, *loc. cit.*, that it was possible to obtain in any space of constant curvature a group analogous to his group of whirls in the plane and that, moreover, for ordinary space the *feuillet*, consisting of a point, line, and plane all incident with one another, would be an appropriate element. Another generalization of Kasner's turbine geometry, along lines different from those pursued in this paper, has been carried out by A. Narasinga Rao, *Studies in turbine geometry* I, Journal of the Indian Mathematical Society, vol. 3 (1938), pp. 96–108; II, Proceedings of the Indian Academy of Sciences, vol. 8A (1938), pp. 179–186.

OR be an axis through the origin parallel to a given unit vector v , and let

$$\zeta = -\cos \theta/2 + v \sin \theta/2, \quad \bar{\zeta} = -\cos \theta/2 - v \sin \theta/2.$$

Then the rotation of z around OR through the angle θ is given by Hamilton's formula⁵

$$(2.1) \quad z^* = \bar{\zeta} z \zeta.$$

We shall refer to (2.1) as the rotation ζ .

Let $\mathfrak{E}$ be a plane element characterized by the point z lying in the plane P and the pair of fundamental orthogonal directions given by the unit vectors v_1, v_2 , in this order. Let ζ be the (unit) quaternion by means of which the vectors e_1 and e_2 can be rotated around an axis passing through the origin into positions parallel respectively to the directions of v_1 and v_2 . There evidently exists a (1, 1) correspondence between the plane elements $\mathfrak{E}$ and the pairs of quantities z, ζ ; we shall, accordingly, call z, ζ the coordinates of $\mathfrak{E}$, and shall use the symbol (z, ζ) to designate the plane element whose coordinates are z, ζ .

DEFINITIONS. A *slide* is a plane element transformation, $(z, \zeta) \rightarrow (z^*, \zeta^*)$, which leaves the planes of the plane elements unaltered, but subjects their points to the same translation relative to their fundamental directions.

A *turn* is a plane element transformation which leaves the points of the plane elements unaltered, but rotates their planes through the same angle around an axis inclined in the same manner relative to their fundamental directions.

A *direct whirl*⁶ is a plane element transformation resulting from the product of an arbitrary turn followed first by an arbitrary slide and then by a second arbitrary turn.

These definitions are the space analogues of those given by Kasner for the plane.

Let $a = a_1 e_1 + a_2 e_2$, (a_i real); then the slide $(z, \zeta) \rightarrow (z^*, \zeta^*)$, which shall be represented by S_a , is given by the equations:

$$(2.2) \quad z^* = z + \bar{\zeta} a \zeta, \quad \zeta^* = \zeta.$$

⁵ It will suffice for our purpose to restrict ourselves throughout this paper to *unit* quaternions—that is, those having a norm equal to unity.

⁶ The term *direct whirl* is used here instead of Kasner's *whirl* because it is necessary to distinguish this from other types of whirl that appear below. This term was first used, for a similar reason, in J. M. Feld's *Whirl-similitudes, euclidean kinematics, and non-euclidean geometry*. (This Bulletin, abstract 46-5-270.)

Let $\alpha = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ and $N(\alpha) = 1$; then the turn T_α is given by the equations:

$$(2.3) \quad z^* = z, \quad \zeta^* = \alpha \zeta.$$

The direct whirl $T_\alpha S_a T_\beta$ is given by the equations:

$$z^* = \bar{\zeta} \bar{\alpha} a \alpha \zeta, \quad \zeta^* = \beta \alpha \zeta.$$

If we let $\bar{\alpha} a \alpha = c$ and $\beta \alpha = \gamma$, the equations of a generic direct whirl $W_{\alpha, \gamma}^+$ assume the form:

$$(2.4) \quad z^* = z + \bar{\zeta} c \zeta, \quad \zeta^* = \gamma \zeta$$

where c is a vector and γ is a unit quaternion.

We can now state the following theorems:

THEOREM 1. *The slides constitute a continuous two-parameter abelian group such that $S_a \cdot S_b = S_{a+b}$.*

THEOREM 2. *The turns constitute a continuous three-parameter group such that $T_\alpha \cdot T_\beta = T_{\beta\alpha}$.*

THEOREM 3. *The direct whirls constitute a continuous six-parameter group. A necessary and sufficient condition that $W_{\alpha, \gamma}^+ = W_{a, \alpha}^+ \cdot W_{b, \beta}^+$ is that $c = a + \bar{a} b \alpha$ and $\gamma = \beta \alpha$.*

We shall let $\mathfrak{B}_6^+$ represent the group of direct whirls. Noting that the direct whirls $W_{a, \alpha}^+$ where $\alpha = e_0$, $a = k e_3$ (k real), form a one-parameter continuous group of contact transformations—namely, the group of dilatations D_k —we obtain the following theorem:

THEOREM 4. *Direct whirls can be uniquely represented in the form $W_{\alpha, \gamma}^+ = D_k \cdot S_b \cdot T_\alpha$ where $c = k e_3 + b$, $\gamma = \alpha$.*

From equations (2.4) we obtain, by multiplying each member of the first equation on the left by the corresponding member of the second equation and on the right by the conjugate of the corresponding member of the second equation, $\zeta^* z^* \bar{\zeta}^* = \gamma (\zeta z \bar{\zeta}) \bar{\gamma} + \gamma c \bar{\gamma}$. Letting $Z^* = \zeta^* z^* \bar{\zeta}^*$, $Z = \zeta z \bar{\zeta}$, $\alpha = \bar{\gamma}$, $a = \gamma c \bar{\gamma}$, we have

$$(2.5) \quad Z^* = \bar{\alpha} Z \alpha + a,$$

which represents a euclidean point displacement $Z \rightarrow Z^*$ in S_3 . Let this displacement be represented by $D_{a, \alpha}^+$. Then we obtain this theorem:

THEOREM 5. *The group of direct whirls is simply isomorphic to the group of euclidean displacements in S_3 : the correspondence between members of these groups is given by $D_{a, \alpha}^+ \leftrightarrow W_{b, \beta}^+$ where $b = \alpha a \bar{\alpha}$ and $\beta = \bar{\alpha}$.*

3. **The group of whirl-motions.** Equation (2.5), which represents a point transformation, can be extended to represent a plane element transformation $(z, \zeta) \rightarrow (z^*, \zeta^*)$ as follows:

$$(3.1) \quad z^* = \bar{\alpha}z\alpha + a, \quad \zeta^* = \zeta\alpha.$$

Henceforth, $D_{a,\alpha}^+$ shall designate the euclidean displacement of plane elements given by (3.1); $\mathfrak{D}_6^+$ shall designate the six-parameter group of these displacements. The product of a displacement $D_{a,\alpha}^+$ and a direct whirl $W_{b,\beta}^+$, in either order, shall be called a *direct whirl-motion*. The equations of the direct whirl-motion $D_{a,\alpha}^+ \cdot W_{b,\beta}^+$ are:

$$(3.2) \quad z^* = \bar{\alpha}(z + \bar{\zeta}b\zeta)\alpha + a, \quad \zeta^* = \beta\zeta\alpha.$$

Evidently $D_{a,\alpha}^+ \cdot W_{b,\beta}^+ = W_{b,\beta}^+ \cdot D_{a,\alpha}^+$. The direct whirl-motions constitute a continuous twelve-parameter group, which shall be designated by $\mathfrak{G}_{12}^1$.

Let l and r be an arbitrary pair of vectors, and let the plane element $\mathfrak{E}: (l+r, e_0)$ be subjected to the ∞^3 possible rotations around the point whose position vector is equal to l ; then the manifold of ∞^3 plane elements (z, ζ) into which $\mathfrak{E}$ is thereby transformed shall be called a (space) *turbine*. The equation of this turbine is

$$(3.3) \quad z - l = \bar{\zeta}r\zeta.$$

Since the vectors l and r completely characterize the turbine (3.3), we shall call l and r its left and right coordinates respectively, and let the symbol $[l, r]$ represent the turbine whose equation is (3.3). A brief computation will verify this theorem:

THEOREM 6. *Turbines are transformed into turbines under $\mathfrak{G}_{12}^1$. Under $\mathfrak{B}_6^+$ their left coordinates remain invariant; under $\mathfrak{D}_6^+$ their right coordinates remain invariant.*

Let us now consider a space analogue of the *flat field*, a concept that appears in Kasner's whirl-motion geometry in the plane. We shall call this analogue a *dual element* because it will serve as the dual of the plane element, just as the flat field serves as the dual of the lineal element.⁷ Dual elements shall be represented by two coordinates s and σ where the former is a vector and the latter is a unit quaternion; in order to distinguish dual elements from plane elements, the former shall be represented by the symbol $\{s, \sigma\}$. We define dual elements by imposing upon them the requirement that the turbine $[l, r]$ which, in plane element coordinates z, ζ , has the equation (3.3) shall, in dual coordinates s, σ , have the equation

⁷ Feld, loc. cit.

$$(3.3^*) \quad s - l = -\bar{\sigma} r \sigma.$$

A turbine $[l, r]$ and a plane element (z, ζ) shall be said to be *incident* if their coordinates are connected by equation (3.3); similarly, $[l, r]$ and $\{s, \sigma\}$ shall be said to be incident if their coordinates are connected by equation (3.3*).

It is evident that direct whirls and displacements of plane elements induce contragredient transformations of the dual elements; for example, the dual element transformation contragredient to $W_{a,\alpha}^+$ is

$$(3.4) \quad s^* = s - \bar{\sigma} a \sigma, \quad \sigma^* = \alpha \sigma.$$

This can be regarded, of course, as a representation of $W_{a,\alpha}^+$ in dual element coordinates; likewise, displacements and whirl-motions can be expressed in dual element coordinates.

Let $\mathfrak{F}_1$ represent the involutory plane element transformation

$$(3.5) \quad z^* = -z, \quad \zeta^* = \zeta,$$

which induces the dual element transformation

$$(3.5^*) \quad s^* = -s, \quad \sigma^* = \sigma.$$

The ∞^6 transformations $\mathfrak{F}_1 \mathfrak{B}_6^+$ or $\mathfrak{B}_6^+ \mathfrak{F}_1$ shall be called *opposite whirls*; the family of opposite whirls shall be designated by $\mathfrak{B}_6^-$. Similarly, the family $\mathfrak{F}_1 \mathfrak{D}_6^+ \equiv \mathfrak{D}_6^+ \mathfrak{F}_1$, which embraces the ∞^6 euclidean symmetries in S_3 , shall be designated by $\mathfrak{D}_6^-$; and the twelve-parameter family yielded by the products of $\mathfrak{F}_1$ and the direct whirl-motions shall be called the family of *opposite whirl-motions*, and shall be represented by $\mathfrak{G}_{12}^2$. Evidently, $\mathfrak{D}_6^- \mathfrak{B}_6^+ \equiv \mathfrak{D}_6^+ \mathfrak{B}_6^- \equiv \mathfrak{G}_{12}^2$. Since the product of two opposite whirl-motions is a direct whirl-motion, the two families $\mathfrak{G}_{12}^1$ and $\mathfrak{G}_{12}^2$ form a mixed group.

Let $\mathfrak{F}_2$ represent the involutory correlation $(z, \zeta) \leftrightarrow \{s, \sigma\}$ given by the equations:

$$(3.6) \quad z = s, \quad \zeta = \sigma.$$

Let the family of transformations embraced by $\mathfrak{F}_2 \mathfrak{G}_{12}^1 \equiv \mathfrak{G}_{12}^1 \mathfrak{F}_2$ be represented by $\mathfrak{G}_{12}^3$; likewise, let $\mathfrak{F}_2 \mathfrak{G}_{12}^2 \equiv \mathfrak{G}_{12}^2 \mathfrak{F}_2$ be represented by $\mathfrak{G}_{12}^4$. The four families $\mathfrak{G}_{12}^i$, ($i=1, 2, 3, 4$), constitute a mixed group, which shall be designated by Γ_{12} and called the *group of proper whirl-motions*.

Let $\mathfrak{F}_3$ represent the involutory plane element transformation $(z, \zeta) \rightarrow (z^*, \zeta^*)$ given by the equations:

$$(3.7) \quad z^* = \zeta \bar{z} \bar{\zeta}, \quad \zeta^* = \bar{\zeta}.$$

This transformation induces the dual element transformation

$$(3.7^*) \quad s^* = -\sigma s \bar{\sigma}, \quad \sigma^* = \bar{\sigma}.$$

Let the products $\mathfrak{S}_3 \mathfrak{G}_{12}^i \equiv \mathfrak{G}_{12}^i \mathfrak{S}_3$, ($i=1, 2, 3, 4$), be designated by $\mathfrak{S}_{12}^i$. The four families $\mathfrak{S}_{12}^i$ embrace the *improper whirl-motions*. The proper and improper whirl-motions constitute a mixed twelve-parameter group composed of eight mutually exclusive families; this group, which shall be designated by $\overline{\Gamma}_{12}$, shall be called the *complete group of whirl-motions*. The following theorems are now evident.

THEOREM 7. *Turbines are transformed into turbines under the complete group of whirl-motions.*

THEOREM 8. *The turbine transformation effected by $\mathfrak{S}_3$ is given by $l^* = r$, $r^* = l$.*

4. The kinematic representation. Let the turbine $[l, r]$ be mapped on that ordered pair of points in S_3 whose position vectors are l and r , in this order. By means of this mapping, which shall be known as the *kinematic representation*,⁸ a (1, 1) correspondence is set up between turbines $[l, r]$ and ordered point pairs l, r , the former of which is the left image point and the latter the right image point of $[l, r]$. If we regard, as we may, the 8×12 whirl-motions in $\overline{\Gamma}_{12}$ as turbine transformations, $[l, r] \rightarrow [l^*, r^*]$, we find that the kinematic representation maps these transformations (1, 1) upon pairs of euclidean motions (displacements and symmetries) in S_3 , which are either of the type $l \rightarrow l^*$, $r \rightarrow r^*$, or else of the type $l \rightarrow r^*$, $r \rightarrow l^*$. The pairs of motions of the former type correspond to the proper whirl-motions, and those of the latter type to the improper whirl-motions. Explicitly, the kinematic images of the families $\mathfrak{G}_{12}^i$ are as follows:

$$(4.1) \quad \begin{aligned} \mathfrak{G}_{12}^1: \quad l^* &= \bar{\alpha}l\alpha + a, & r^* &= \bar{\beta}r\beta + b, \\ \mathfrak{G}_{12}^2: \quad l^* &= -\bar{\alpha}l\alpha + a, & r^* &= -\bar{\beta}r\beta + b, \\ \mathfrak{G}_{12}^3: \quad l^* &= \bar{\alpha}l\alpha + a, & r^* &= -\bar{\beta}r\beta + b, \\ \mathfrak{G}_{12}^4: \quad l^* &= -\bar{\alpha}l\alpha + a, & r^* &= \bar{\beta}r\beta + b, \end{aligned}$$

α and β are unit quaternions; a and b are vectors.

To obtain the kinematic image of the family $\mathfrak{S}_{12}^i$ we need but, in ac-

⁸ A similar representation of whirl-motion geometry in the plane has been given by Feld, loc. cit. See also, in this connection, J. Grünwald's *Ein Abbildungsprinzip, welches die ebene Geometrie und Kinematik mit der räumlichen Geometrie verknüpft*, Sitzungsberichte der Akademie der Wissenschaften, Vienna, Mathematisch-Naturwissenschaftliche Klasse (II A), vol. 80 (1911), pp. 677-741; and two contributions by W. Blaschke, *Euklidische Kinematik und nichteuklidische Geometrie*, Zeitschrift für Mathematik und Physik, vol. 60 (1911), pp. 61-91, 203-204; and *Ebene Kinematik*, Berlin, 1938.

cordance with Theorem 8, interchange l and r in the pair of equations given by (4.1) which corresponds to $\mathcal{G}_{13}^4$.

Inasmuch as the turbine $[l, r]$ has for its equation in plane element coordinates $z, \zeta: l = -\bar{\zeta}r\zeta + z$, and for its equation in dual element coordinates $s, \sigma: l = \bar{\sigma}r\sigma + s$, it is apparent that a necessary and sufficient condition that $[l, r]$ be incident to a given plane element (z, ζ) [dual element $\{s, \sigma\}$] is that the left and right image points of $[l, r]$ in the kinematic representation be corresponding points in a euclidean symmetry [displacement]. Consequently, the kinematic representation establishes a (1, 1) correspondence between plane elements [dual elements] and euclidean symmetries [displacements] in S_3 .

The kinematic mapping of plane elements and dual elements upon symmetries and displacements, respectively, furnishes us with the means of representing parametrically the composition of euclidean motions in S_3 .⁹ An example will suffice to show how this can be done. Let the coordinates of the dual element $\{s, \sigma\}$ represent the displacement that corresponds to $\{s, \sigma\}$ in the kinematic representation; let the displacement corresponding to $\{s_3, \sigma_3\}$ be equivalent to the product of the displacements corresponding to $\{s_1, \sigma_1\}$ and $\{s_2, \sigma_2\}$, and in this order. With these conventions it follows that

$$s_3 = \bar{\sigma}_2 s_1 \sigma_2 + s_2, \quad \sigma_3 = \sigma_1 \sigma_2;$$

in a similar manner we can represent other combinations of euclidean motions.

5. The group of whirl-similitudes. Let $\mathcal{S}$ represent the pair of similitudes:

$$l^* = kl, \quad r^* = kr$$

where k is a positive real number, and l, r are the kinematic image points of $[l, r]$. Let $\mathcal{G}_{13}^i \equiv \mathcal{S}\mathcal{G}_{12}^i$ and $\mathcal{H}_{13}^i \equiv \mathcal{S}\mathcal{H}_{12}^i$. Then the eight thirteen-parameter families comprising $\mathcal{G}_{13}^i$ and $\mathcal{H}_{13}^i$, ($i=1, 2, 3, 4$), constitute a mixed group $\bar{\Gamma}_{13}$ —the *complete group of whirl-similitudes*. This group is the space analogue of the group of whirl-similitudes in the plane.¹⁰ Evidently, turbines are transformed into turbines under $\bar{\Gamma}_{13}$.

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⁹ E. Study has given such a representation by means of Clifford biquaternions, op. cit.

¹⁰ Feld, op. cit.

A THEOREM CONCERNING THE GEODESICS ON A PARABOLOID OF REVOLUTION¹

DONALD LING AND LEON RECHT

We shall examine the behavior in the large of the geodesics on a paraboloid of revolution. It will appear that on every geodesic there is a unique point—the “point of symmetry”—which divides the geodesic into two “conjugate rays,” which are mirror images of each other in the plane determined by that point and the axis of the paraboloid. Every geodesic except the meridians, which have no singularities whatever, has infinitely many double points, but no other singular points. If, starting at the point of symmetry, we imagine the conjugate rays of a geodesic to be traversed simultaneously by two points which are images in the plane of symmetry, and which continue until they meet for the first time—at the “first double point” of the geodesic—we shall have generated a “loop of type 1.” If, starting at the point of symmetry, they pass through the first double point and continue until they next meet—at the “second double point”—we shall have formed a “loop of type 2.” We define in this way loops of types 1, 2, 3, ... The last double point to be reached in the process of generating a loop of a given type will be called the “vertex” of the loop.

The main result of this paper is the following: The paraboloid can be divided by planes perpendicular to the axis into infinitely many zones, each zone to include the points of that bounding circle which is nearer the vertex. The zone containing the vertex we call the “cap,” the following the “first zone,” and so on. The zones have the following properties:

1. Each interior point of the k th zone is the vertex of exactly two distinct loops of each of the types 1, 2, . . . , k , but is the vertex of no loop of type greater than k .
2. Each boundary point in the k th zone is the vertex of two loops of each of the types 1, 2, . . . , $k-1$ and of one loop of type k . It is the vertex of no loop of type greater than k .
3. No point of the cap is the vertex of any loop.

Let the paraboloid be that obtained by revolving $r^2 = x$ about the x -axis. As coordinates of a point P on the surface (the vertex excepted) we take r to be the length of the vector R from P perpendicular to the axis and directed toward P , and θ to be the angle which the plane determined by P and the axis makes with an arbitrary plane contain-

¹ Presented to the Society, February 24, 1940.

ing the axis. The differential equation defining the geodesics (cf. Darboux, *Leçons sur la Théorie Générale des Surfaces* . . . , vol. 3) is

$$(1) \quad \frac{d\theta}{dr} = -\frac{c}{r} \left[\frac{4r^2 + 1}{r^2 - c^2} \right]^{1/2}$$

where c is an arbitrary constant. Corresponding to $c=0$ are the meridians $\theta=\text{const}$. Since their nature is obvious we may except them from consideration, and assume $c \neq 0$. The solution of (1) corresponding to a given value of $c \neq 0$ and passing through (r_0, θ_0) is

$$(2) \quad \begin{aligned} \theta = \theta_0 + \arctan c \left[\frac{4r^2 + 1}{r^2 - c^2} \right]^{1/2} - c \log \frac{(4r^2 + 1)^{1/2} + 2(r^2 - c^2)^{1/2}}{(4r^2 + 1)^{1/2} - 2(r^2 - c^2)^{1/2}} \\ - \arctan c \left[\frac{4r_0^2 + 1}{r_0^2 - c^2} \right]^{1/2} + c \log \frac{(4r_0^2 + 1)^{1/2} + 2(r_0^2 - c^2)^{1/2}}{(4r_0^2 + 1)^{1/2} - 2(r_0^2 - c^2)^{1/2}} \\ = \theta_0 + f(r; r_0, c), \text{ say.} \end{aligned}$$

As $r \rightarrow |c|$ the right member of (2) approaches

$$(3) \quad \begin{aligned} \bar{\theta} = \theta_0 + \frac{\pi}{2} \text{sign } c - \arctan c \left[\frac{4r_0^2 + 1}{r_0^2 - c^2} \right]^{1/2} \\ + c \log \frac{(4r_0^2 + 1)^{1/2} + 2(r_0^2 - c^2)^{1/2}}{(4r_0^2 + 1)^{1/2} - 2(r_0^2 - c^2)^{1/2}}. \end{aligned}$$

It can be shown without difficulty that $(|c|, \bar{\theta})$ is a point of the corresponding geodesic.³ It is apparent from (2) and (3) that if $c < 0$ θ increases monotonically from $\bar{\theta}$ at $r = |c|$ to ∞ at $r = \infty$, while if $c > 0$ θ decreases monotonically from $\bar{\theta}$ at $r = |c|$ to $-\infty$ at $r = \infty$.

Consider the two solutions

$$\begin{aligned} \theta^{(1)} &= \theta_0 + f(r; r_0, c), \\ \theta^{(2)} &= \{\theta_0 + 2f(|c|; r_0, c)\} + f(r; r_0, -c) \\ &= \{\theta_0 + 2f(|c|; r_0, c)\} - f(r; r_0, c). \end{aligned}$$

It is clear that $\theta^{(1)}(|c|) = \theta^{(2)}(|c|) = \bar{\theta}$. Moreover, at the point $(|c|, \bar{\theta})$ the two solutions have a common tangent, and thus belong to the same geodesic. Furthermore

$$(4) \quad \theta^{(1)}(r) - \bar{\theta} = f(r; r_0, c) - f(|c|; r_0, c) = -[\theta^{(2)}(r) - \bar{\theta}]$$

³ It is readily verified that if r and θ are expressed in terms of the arc length s along the geodesic measured from (r_0, θ_0) , then $r \rightarrow |c|$ and $\theta \rightarrow \bar{\theta}$ as s approaches a certain finite limit.

so that the points $(r, \theta^{(1)}(r))$ and $(r, \theta^{(2)}(r))$ are mirror images of each other in the plane determined by the axis and the point $(|c|, \bar{\theta})$. This is the point of symmetry of the geodesic, and equations (4) give a pair of conjugate rays as defined above. Bearing in mind that $\theta^{(1)}(r)$ and $\theta^{(2)}(r)$ are monotonic in opposite senses, and that the absolute value of each is large with r , we see that each geodesic has infinitely many double points, but no other multiple points.

Assume $\theta_0 = 0$. Then $\bar{\theta}$ is the angle through which the radius vector R turns as (r, θ) moves along a geodesic ray from its initial point $(r_0, 0)$ to its point of symmetry $(|c|, \bar{\theta})$. We shall study the variation of $\bar{\theta}$ as the parameter c , which controls the direction of the geodesic ray at its initial point $(r_0, 0)$, varies over non-negative values.

We find from (3) for $c > 0$ that

$$(5) \quad \frac{d\bar{\theta}}{dc} = -\lambda + \log \frac{\lambda + 2}{\lambda - 2}; \quad \lambda = \left[\frac{4r_0^2 + 1}{r_0^2 - c^2} \right]^{1/2}.$$

As λ increases from 2 to ∞ the logarithm in (5) decreases monotonically from ∞ to 0. Thus $d\bar{\theta}/dc = 0$ has a single root $\lambda_0 > 2$. In terms of c , from the second of equations (5), $d\bar{\theta}/dc$ vanishes when and only when $c = c^* = ((\lambda_0^2 - 4)r_0^2 - 1)^{1/2}$. This root is real if, as we now assume, $r_0 \geq 1/(\lambda_0^2 - 4)^{1/2}$. If $c > c^*$, $d\bar{\theta}/dc < 0$; if $c < c^*$, $d\bar{\theta}/dc > 0$. Since $\bar{\theta}(r_0, c)$ is continuous in c for $0 \leq c < r_0$, and since $\bar{\theta}(r_0, 0) = \pi/2$ and $\lim_{c \rightarrow r_0} \bar{\theta}(r_0, c) = 0$, $\bar{\theta}(r_0, c)$ attains its greatest value, $\theta^*(r_0)$, at $c = c^*$.

From (3) and (5)

$$(6) \quad \theta^*(r_0) = \pi/2 - \arctan \lambda_0((\lambda_0^2 - 4)r_0^2 - 1)^{1/2} \\ + \lambda_0((\lambda_0^2 - 4)r_0^2 - 1)^{1/2}.$$

This function is continuous for $r_0 \geq 1/(\lambda_0^2 - 4)^{1/2}$ and its derivative, $((\lambda_0^2 - 4)r_0^2 - 1)^{1/2}/r_0$, is positive for $r_0 \geq 1/(\lambda_0^2 - 4)^{1/2}$. Thus $\theta^*(r_0)$, which gives $\bar{\theta}$ for the geodesic ray of greatest $\bar{\theta}$ through $(r_0, 0)$, is continuous and strictly increasing. Moreover $\theta^*(1/(\lambda_0^2 - 4)^{1/2}) = \pi/2$ and $\lim_{r_0 \rightarrow \infty} \theta^*(r_0) = \infty$. There is, then, a sequence of values of r_0 : $r_0^{(1)} < r_0^{(2)} < r_0^{(3)} < \dots$ such that $\theta^*(r_0^{(k)}) = k\pi$ ($k = 1, 2, \dots$). The portion of the paraboloid defined by $r_0^{(k)} \leq r_0 < r_0^{(k+1)}$ we shall call the " k th zone," and the parallel $r_0 = r_0^{(k)}$ its boundary. Let r_0 be an interior point of the k th zone; then $k\pi < \theta^*(r_0) < (k+1)\pi$. As c increases from 0 to c^* , $\bar{\theta}(r_0, c)$ increases from $\pi/2$ to $\theta^*(r_0)$, and as c then increases from c^* to r_0 $\bar{\theta}(r_0, c)$ decreases from $\theta^*(r_0)$ to 0. Hence there are two sequences of values of c , $\{c_1^{(1)}, \dots, c_k^{(1)}\}$ and $\{c_1^{(2)}, \dots, c_k^{(2)}\}$ satisfying $c_1^{(1)} < c_2^{(1)} < \dots < c_k^{(1)} < c^* < c_k^{(2)} < \dots < c_1^{(2)}$ and having the property that $\bar{\theta}(r_0, c_m^{(1)}) = \bar{\theta}(r_0, c_m^{(2)}) = m\pi$, ($m = 1, \dots, k$). When r_0 is

on the boundary of the k th zone, $c_k^{(1)} = c^* = c_k^{(2)}$; but, for $m < k$, $c_m^{(1)}$ and $c_m^{(2)}$ are distinct.

Interpreted geometrically this means that through a point $(r_0, 0)$ in the k th zone there are, for each value of $m = 1, \dots, k$, two geodesic rays for which $\bar{\theta} = m\pi$. The conjugate ray of each, being a reflection of the original ray, clearly passes through $(r_0, 2m\pi)$, which is identical with $(r_0, 0)$. This point is therefore the vertex of two loops of each of the types $1, \dots, k$. If $(r_0, 0)$ is on the boundary of the k th zone this statement must be modified to the effect that the loops of type k are coincident, those of lower types remaining distinct.

It remains to examine the cap, that is, that portion of the surface for which $r_0 < r_0^{(1)}$. If $1/(\lambda_0^2 - 4)^{1/2} \leq r_0 < r_0^{(1)}$, $\theta^*(r_0) < \pi$ and no loop with vertex $(r_0, 0)$ is possible. If $r_0 < 1/(\lambda_0^2 - 4)^{1/2}$, then $\lambda = [(4r_0^2 + 1)/(r_0^2 - c^2)]^{1/2} \geq (4 + 1/r_0^2)^{1/2} > \lambda_0$ and $d\bar{\theta}/dc < 0$. As c increases from 0 to r_0 , $\bar{\theta}(r_0, c)$ decreases from $\pi/2$ to 0. Thus no loop having a vertex in this portion is possible. We have now established all the results stated in the introduction.

The numbers $r_0^{(n)}$ can be calculated from (6). For, if we set $\theta^*(r_0) = n\pi$ and replace $(\lambda_0^2 - 4)r_0^2 - 1$ by μ , (6) becomes $\mu - \arctan \mu = (2k - 1)\pi/2$. Since the left member increases monotonically from 0 to ∞ as μ increases from 0 to ∞ , there is exactly one root μ_n for each value of $r = 1, 2, 3, \dots$, and $r_0^{(n)} = [(\mu_n^2 + 1)/(\lambda_0^2 - 4)]^{1/2}$. For large values of n , μ_n is of the order of $n\pi$, and $r_0^{(n)}$ is of the order of $[(n^2\pi^2 + 1)/(\lambda_0^2 - 4)]^{1/2}$. We have approximately $\lambda_0 = 2.40$ and $1/(\lambda_0^2 - 4)^{1/2} = 3/4$. Thus $3n\pi/4$ furnishes an estimate of $r_0^{(n)}$ for large values of n .

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PROJECTION OF THE SPACE (m) , ON ITS SUBSPACE (c_0)

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In a paper in the Duke Journal, A. E. Taylor¹ remarks that it is an open question whether or not there exists a projection of the space (m) , of bounded sequences, on its subspace (c_0) , the space of sequences convergent to 0. In this note we make a few remarks which supplement those of Taylor on this question, and we point out that a negative answer follows from a recent result of R. S. Phillips,² so that the question is now settled.

Taylor shows that if a projection of the space (c) , of convergent sequences, on the space (c_0) exists, it must be of norm greater than or equal to 2. This implies the same result for (m) on (c_0) , since any projection of (m) on (c_0) would be in particular a projection of (c) on (c_0) .

The space (c) is essentially of dimension only one greater than that of its subspace (c_0) . This follows since (c) is obviously the set of all elements of the form $x = x^{(0)} + tX_1$, where $X_1 = (1, 1, \dots)$, $x^{(0)} \in (c_0)$, and t is a number. If $x = \{x_i\}$ is any element of (c) , the linear functional $a(x) = t = \lim_{n \rightarrow \infty} x_n$ is of norm 1, and vanishes on the subspace (c_0) . Now it is a remark of Bohnenblust³ that for any subspace of a normed linear space L defined by the vanishing of a fixed linear functional on L , there exist projections of norm less than or equal to $2 + \epsilon$, for arbitrary $\epsilon > 0$. Consequently there are projections of (c) on (c_0) of norm less than or equal to $2 + \epsilon$.

There are projections of (c) on (c_0) which are of norm exactly 2, as may be seen as follows. If $x = (x^{(0)} + tX_1) \in (c)$, the general form of a projection of (c) on (c_0) is

$$Px = x + t\{b_i\} = x^{(0)} + t(X_1 + \{b_i\})$$

where $\{b_i\}$ is any sequence of constants such that $\lim_{i \rightarrow \infty} b_i = -1$.⁴ To calculate the norm of P , we have $\|Px\| \leq \|x\| + |t| \cdot \sup_i |b_i|$, and $\|x\| = \|x^{(0)} + tX_1\| = \sup_i |x_i^{(0)} + t| \geq |t|$ since $x_i^{(0)} \rightarrow 0$. Therefore $\|P\| \leq 1 + \sup_i |b_i|$, and because of Taylor's result this has the value

¹ *The extension of linear functionals*, Duke Mathematical Journal, vol. 5 (1939), pp. 538-547; p. 547.

² *On linear transformations*, Transactions of this Society, vol. 48 (1940), pp. 516-541; pp. 539-540.

³ *Convex regions and projections in Minkowski spaces*, Annals of Mathematics, (2), vol. 39 (1938), pp. 301-308; p. 308.

⁴ See Taylor, op. cit.

2 for all projections such that $|b_i| \leq 1$ for all i . In particular the projection $Px = x - a(x) \cdot X_1 = x^{(0)}$, obtained by taking $b_i = -1$ for all i , is of norm 2.

Corresponding to any sequence $\{e_i\}$, where, for each i , e_i is either $+1$ or -1 , let us define a space $c_{\pm} \dots$ as the set of all elements of the form $x + t\{e_i\}$, where $x \in (c_0)$. Any space $c_{\pm} \dots$ is equivalent⁵ to (c) under the automorphism $\{x_i\} \leftrightarrow \{e_i x_i\}$ of (m) . Also (c_0) is a subspace of $c_{\pm} \dots$, since (c_0) is invariant under the automorphism. There are obviously projections of norm 2 of any $c_{\pm} \dots$ on (c_0) , similar to the projections of (c) on (c_0) .

If L and L' are any two linear subspaces of (m) , we define the *sum* $L + L'$ as the linear subspace of (m) which consists of all elements of the form $x + x'$, where $x \in L$, $x' \in L'$. As usual we denote the closure in (m) of any subspace by a horizontal line over the symbol representing the subspace.

For any set of n linearly independent sequences $X_j = \{e_{ij}\}$, $e_{ij} = +1$ or -1 , $j = 1, 2, \dots, n$, let l_n denote the n dimensional space of all linear combinations $\sum_j \alpha_j X_j$. Suppose the X_j 's are such that l_n and (c_0) do not intersect, except in the origin. Then we may define a subspace (c_n) of (m) as the space $(c_0) + l_n$. Similarly, for any countable sequence of X_j 's, any finite number of which are linearly independent, we denote the space of all finite linear combinations $\sum_j \alpha_j X_j$ by l_f , and we define $l_{\infty} = \overline{l_f}$ ($\infty = \aleph_0$). If l_f and (c_0) intersect only in the origin, we define

$$(c_{\infty}) = \overline{(c_0) + l_{\infty}} = \overline{(c_0) + l_f}.$$

(If l_{∞} and (c_0) intersect only in the origin, each $x \in (c_0) + l_{\infty}$ has a unique decomposition $x = x^{(0)} + x^{(\infty)}$, $x^{(0)} \in (c_0)$, $x^{(\infty)} \in l_{\infty}$; otherwise not. The space $(c_0) + l_{\infty}$ is not necessarily closed.)

Let $X_1 = \{e_{i1}\}$, $X_2 = \{e_{i2}\} \dots$ be the rows of an infinite matrix $\{\{e_{ij}\}\}$; let $\{\{e_{ij}\}\}_n$ denote the matrix of the first n rows of $\{\{e_{ij}\}\}$. Then any space (c_n) or (c_{∞}) is determined by a matrix $\{\{e_{ij}\}\}_n$ or $\{\{e_{ij}\}\}$.

THEOREM 1. *In any space (c_n) [or (c_{∞})] such that for every column C_k which appears in the matrix $\{\{e_{ij}\}\}_n$ [for each n] one of $+C_i$, $-C_i$ is identical with C_k for an infinite number of values of i , as $i \rightarrow \infty$, there exists a projection of (c_n) or (c_{∞}) on (c_0) , which is of norm 2.*

⁵ Two normed linear spaces are isomorphic if there exists a 1-1 transformation T between them which is linear (that is, distributive and continuous) in both directions; they are equivalent if in addition $|T| = |T^{-1}| = 1$.

PROOF. If $x = x^{(0)} + \sum_{j=1}^n t_j X_j$, $x^{(0)} \in (c_0)$, define $Px = x^{(0)}$. Then

$$\|Px\| = \left\| x - \sum_j t_j X_j \right\| \leq \|x\| + \left\| \sum_j t_j X_j \right\| = \|x\| + \sup_i \left| \sum_j t_j e_{ij} \right|.$$

By the hypothesis on the matrix $\{\{e_{ij}\}\}_n$, and since if $x^{(0)} = \{x_i^{(0)}\}$, $x_i^{(0)} \rightarrow 0$, $\|x\| = \sup_i |x_i^{(0)} + \sum_j t_j e_{ij}| \geq \max_i |\sum_j t_j e_{ij}|$. This implies $\|Px\| \leq 2\|x\|$ for any x of (c_n) or of the dense linear set in (c_∞) . In the latter case P has a unique continuous extension to (c_∞) , with range (c_0) , and the norm is preserved. This verifies the theorem.

THEOREM 2. *For every space (c_n) , $1 \leq n \leq \infty$, there is a projection of (c_n) on (c_0) which is of norm 2.*

PROOF. Let (c_n) be determined by $X_1 = \{e_{i1}\}$, $X_2 = \{e_{i2}\}$, $\dots$, as above. Consider first the case of any finite n . Since only a finite number of different n -element columns are possible, there can be only a finite number of columns C_k in the matrix $\{\{e_{ij}\}\}_n$ which are not infinitely repeated. Since X_1 and X_2 are linearly independent, and l_2 intersects (c_0) only in the origin, the matrix $\{\{e_{ij}\}\}_2$ evidently satisfies the hypothesis of Theorem 1. Consider $(c_3) = (c_0) + l_3$. By altering at most a finite number of coordinates of X_3 , the matrix $\{\{e_{ij}\}\}_3$ can be made to satisfy the hypothesis. Let the altered sequence X_3 be X'_3 ; then $X'_3 = X_3 + y^{(0)}$, where $y^{(0)} \in (c_0)$. Therefore the space $(c_3)'$ determined by X_1, X_2, X'_3 coincides with (c_3) . Proceeding in similar fashion, we see that any space (c_n) or $(c_0) + l_r$ may be represented by a matrix $\{\{e_{ij}\}\}_n$ or $\{\{e_{ij}\}\}$ which *does* satisfy the hypothesis of Theorem 1. Our theorem is thus proved.

THEOREM 3. *There exists a matrix $E = \{\{e_{ij}\}\}$, such that: (1) the corresponding subspace l_∞ of (m) intersects (c_0) only in the origin, and is equivalent to the space $l_{1,\infty}$ of absolutely convergent series; (2) $(c_0) + l_\infty$ is closed in (m) , and l_∞ and (c_0) are complementary subspaces in $(c_\infty) = (c_0) + l_\infty$; (3) the projection of (c_∞) on (c_0) according to the complementary subspace l_∞ of (c_0) in (c_∞) is of norm 2.*

PROOF. Let $+$ and $-$ represent $+1$ and -1 . The matrix E is

$$\begin{array}{cccccccc} + & - & + & - & + & - & + & - \dots \\ + & + & - & - & + & + & - & - \dots \\ + & + & + & + & - & - & - & - \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}$$

(The k th row of E consists of alternately 2^{k-1} plus signs and 2^{k-1} minus signs, for all k .) In this matrix, obviously *all* possible finite

combinations of + 's and - 's occur in the columns. Thus we have for any n $\|\sum_1^n t_j X_j\| = \sum_1^n |t_j|$. The space l_∞ determined by E is therefore equivalent to $l_{1,\infty}$. Suppose that (c_0) and (l_∞) have a common element x . Then there is a sequence $\{x_n\}$, $x_n = \sum_1^n c_{j,n} X_j$, which converges to x . Since $x \in (c_0)$, by the construction of E we must have $\sum_1^n |c_{j,n}| \rightarrow 0$ as $n \rightarrow \infty$. But this implies $x_n \rightarrow 0$, $x = 0$. This verifies statement (1) of Theorem 3; statement (3) follows by Theorem 1. Statement (2) then follows by a lemma of F. J. Murray, that (for Banach spaces) the existence of projections and of complementary manifolds are equivalent.⁶

THEOREM 4. *There exists a subspace of (m) equivalent to the space (C) of continuous functions, such that: (1) the subspace (C) of (m) and (c_0) intersect only in the origin; (2) $(C) + (c_0)$ is closed in (m) ; (3) the projection of $(C) + (c_0)$ on (c_0) according to the complementary subspace (C) of (c_0) in $(C) + (c_0)$ is of norm 2.*

PROOF. Let $\{r_i\}$ be the set of all rational numbers of the interval $(0, 1)$, in their usual enumeration. Then any $x = x(t)$ of (C) is uniquely determined by the set of values $\{x(r_i)\}$, while (m) may be regarded as the set of all bounded real-valued functions $x = \{x_i\} = \{x(r_i)\}$ on the set $\{r_i\}$. Clearly this correspondence of each $x(t) \in (C)$ to its set of values $\{x(r_i)\}$ in (m) is an equivalence. Furthermore it is obvious that if $\lim_{i \rightarrow \infty} x(r_i) = 0$, $x(t) \equiv 0$. Therefore (C) according to this imbedding intersects (c_0) only in 0.

If $x = \{x(r_i)\}$ corresponds to $x(t) \in (C)$, and $x^{(0)} = \{x^{(0)}(r_i)\}$ belongs to the subspace (c_0) of (m) , then $\sup_i |x(r_i) + x^{(0)}(r_i)| \geq \max |x(t)| = \sup_i |x(r_i)|$ since $x^{(0)}(r_i) \rightarrow 0$, and $x(t)$ is continuous. Thus $\|x + x^{(0)}\| \geq \|x\|$, and the operation defined by $Q(x + x^{(0)}) = x$ is a projection of norm 1 of $(C) + (c_0)$ on (C) . The operation $P = I - Q$ is a projection of $(C) + (c_0)$ on (c_0) , and $|P| = |I - Q| \leq |I| + |Q| = 2$. The proof of statement (2) of Theorem 4 is the same as the first part of the proof of Theorem 6, below.

It is a well known theorem of Banach that every separable Banach space is equivalent to a subspace of (C) .⁷ Therefore by Theorem 4, if (B) is any separable Banach space, there is a subspace of (m) equivalent to (B) , such that the projection of $(c_0) + (B)$ on (c_0) through (B) is of norm 2.

⁶ See F. J. Murray, Transactions of this Society, vol. 41 (1937), pp. 138-139.

In a normed linear space L , two linear subspaces L_1 and L_2 are called *complementary* manifolds or subspaces if (1) they are closed in L , and intersect only in 0; (2) every $x \in L$ has a decomposition $x = x_1 + x_2$, where $x_1 \in L_1$, $x_2 \in L_2$. It follows by assumption that this decomposition is unique.

⁷ S. Banach, *Opérations Linéaires*, p. 185, Theorem 9.

The result of Theorem 2 may be generalized, by removing the restriction that the e_{ij} 's may have only the values $+1$ or -1 . Let $X_1 = \{e_{i1}\}$ be any element of $(m) - (c_0)$. Let $S_1 = [tX_1]$ be the space of all elements tX_1 , and let $B_1 = (c_0) + S_1$. If P is a projection of B_1 on (c_0) , define a sequence of functionals $\{y_i(x)\}$ by $Px = x^{(0)} + tX_1 + \{y_i(x)\}$, where $x = x^{(0)} + tX_1$, $x^{(0)} \in (c_0)$. The functionals $y_i(x)$ are obviously linear, and $y_i(x^{(0)}) = 0$ for all i and $x^{(0)} \in (c_0)$. Let $y_i(X_1) = a_{i1}$. Then $tPX_1 = tX_1 + t\{a_{i1}\} \in (c_0)$, and we see that the general form of a projection of B_1 on (c_0) is $Px = x^{(0)} + t\{a_{i1} + e_{i1}\}$, where $\{a_{i1}\}$ is any sequence of constants such that $\lim_{i \rightarrow \infty} (a_{i1} + e_{i1}) = 0$.

If $X_1 = \{e_{i1}\}$ as above, let $\limsup_i |e_{i1}| = a$. Define $X'_1 = \{e'_{i1}\}$ by $e'_{i1} = e_{i1}$ if $|e_{i1}| \leq a$, $e'_{i1} = a \operatorname{sign} e_{i1}$ if $|e_{i1}| \geq a$. Then $(X_1 - X'_1) = \{e_{i1} - e'_{i1}\} \in (c_0)$, and the space $B'_1 = (c_0) + S'_1$ is the same as B_1 . Thus without loss of generality we may always assume $|e_{k1}| \leq a = \limsup_i |e_{i1}|$ for all k . This assumption is made in the following paragraph.

There are projections of any space B_1 on (c_0) which are of norm 2. For if $Px = x + t\{a_{i1}\}$, then $\|Px\| \leq \|x\| + |t| \cdot \sup_i |a_{i1}|$. Since $\|x\| = \|x^{(0)} + t\{e_{i1}\}\| = \sup_i |x^{(0)}_i + te_{i1}| \geq \sup_i |te_{i1}|$, we see that $|P|$ is 2 for all P such that $|a_{i1}| \leq |e_{i1}|$ for all i . (Note that in no case can $|P|$ be less than 2.)

For any finite or countable set of linearly independent sequences $X_j = \{e_{ij}\}$, (e_{ij} 's not restricted to ± 1), let S_n or S_f respectively denote the subspace of all linear (finite) combinations $\sum_j t_j X_j$. As before, suppose that S_n or S_f does not intersect (c_0) , except in the origin. Define $B_n = (c_0) + S_n$, $B_f = (c_0) + S_f$, $B_\infty = \overline{B_f}$. The general form of a projection of B_n on (c_0) is $Px = x + \{\sum_j t_j a_{ij}\}$ where $x = x^{(0)} + \sum_j t_j X_j$, $x^{(0)} \in (c_0)$, and the a_{ij} 's are any constants such that $\lim_{i \rightarrow \infty} (a_{ij} + e_{ij}) = 0$ for each j . This follows by additivity of P , since P must be in particular a projection of each of the subspaces $(c_0) + [tX_j]$ of B_n on (c_0) .

THEOREM 5. *If W is any separable Banach subspace of (m) such that $W \supset (c_0)$, there is a projection of W on (c_0) which is of norm 2.*

PROOF. It follows easily from the hypothesis that W is separable; that there exists a sequence $\{X_j\}$ such that either W is a space B_n (n finite), or

$$W = B_\infty = \overline{(c_0) + S_f}.$$

(Conversely, any space B_n or B_∞ is separable.)

Consider any space B_n or B_f determined by a matrix $E = \{\{e_{ij}\}\}$ which is such that the e_{ij} 's have only a finite number N of different values. By an argument similar to the proofs of Theorems 1 and 2,

we see that there is a projection of B_n or B_f on (c_0) which is of norm 2. This projection is specifically the one given by the general form above when $a_{ij} = -e_{ij}$ for all i, j (after the e_{ij} 's have been suitably altered).

For any space B_n or B_f , without loss of generality we may assume that $\|X_j\| = 1$ for all j . For each integer N , divide the interval $0 \leq u \leq 1$ into N subintervals $0 \leq u \leq N^{-1}$, $N^{-1} < u \leq 2N^{-1}$, $\dots$, $(k-1)N^{-1} < u \leq kN^{-1}$, $\dots$, $(N-1)N^{-1} < u \leq 1$. If $|e_{ij}|$ lies in the k th of these intervals, let $k_{ij} = kN^{-1} \text{ sign } e_{ij}$. In this way we associate with the matrix $\{e_{ij}\}_n$ of B_n a matrix $\{k_{ij}\}_n$, in which the k_{ij} 's have only the values $\pm kN^{-1}$, $k=1, \dots, N$. By a finite number of alterations, the matrix $\{k_{ij}\}_n$ may be changed into a matrix $\{k_{ij}^{(N)}\}_n$ in which every column is infinitely repeated; this requires addition of integral multiples of $\pm N^{-1}$ to a finite number of the k_{ij} 's. Alter the matrix $\{e_{ij}\}_n$ by adding the same multiples of $\pm N^{-1}$ to the corresponding e_{ij} 's; we thus obtain a new matrix $\{e_{ij}^{(N)}\}_n$ which represents the same space B_n , and which has $\{k_{ij}^{(N)}\}_n$ for its associated matrix. Let the rows of $\{e_{ij}^{(N)}\}$ be denoted by $\{X_j^{(N)}\}$; those of $\{k_{ij}^{(N)}\}$ by $\{Y_j^{(N)}\}$.

If $x \in B_n$, for each N let $x = x^{(0,N)} + \sum_{j=1}^n t_j X_j^{(N)}$, where $x^{(0,N)} \in (c_0)$. Let $x^{(0,N)} = P^{(N)}x$. Let N take on only the succession of values 2^r , $r=1, 2, 3, \dots$. Then evidently we may make the required alterations in the matrices $\{k_{ij}^{(N)}\}_n$ (and $\{e_{ij}^{(N)}\}_n$) in such a way that $\lim_{N, N'} \|X_j^{(N)} - X_j^{(N')}\| = 0$ for each j ; that is, $\{X_j^{(N)}\}$ is a Cauchy sequence for each j . If $x = x^{(0,N)} + \sum_{j=1}^n t_j X_j^{(N)}$ and $x = x^{(0,N')} + \sum_{j=1}^n t_j X_j^{(N')}$,⁸ then $0 = x^{(0,N)} - x^{(0,N')} + \sum_{j=1}^n t_j (X_j^{(N)} - X_j^{(N')})$, and

$$\|x^{(0,N)} - x^{(0,N')}\| = \left\| \sum_{j=1}^n t_j (X_j^{(N)} - X_j^{(N')}) \right\|.$$

Therefore for each $x \in B_n$, $\{x^{(0,N)}\}$ is a Cauchy sequence of elements of (c_0) .

Let $x^{(0)} = \lim_N x^{(0,N)}$; and define $Px = x^{(0)}$. Then

$$\begin{aligned} \|P^{(N)}x\| &= \left\| x - \sum_{j=1}^n t_j X_j^{(N)} \right\| = \left\| x - \sum_{j=1}^n t_j Y_j^{(N)} + \sum_{j=1}^n t_j (Y_j^{(N)} - X_j^{(N)}) \right\| \\ &\leq \|x\| + \left\| \sum_{j=1}^n t_j Y_j^{(N)} \right\| + \left\| \sum_{j=1}^n t_j (Y_j^{(N)} - X_j^{(N)}) \right\|. \end{aligned}$$

We also have

⁸ The t_j 's do not change with N , because in the expression of any x , for each succeeding N the finite alterations can be compensated by changing $x^{(0,N)}$ —and for each system $\{X_j^{(N)}\}$, the expression of x is unique (by the hypothesis that S_n intersects (c_0) only in the origin).

$$\begin{aligned}
\|x\| &= \left\| x^{(0,N)} + \sum_i t_i X_i^{(N)} \right\| \\
&\geq \left\| x^{(0,N)} + \sum_i t_i Y_i^{(N)} \right\| - \left\| \sum_i t_i (Y_i^{(N)} - X_i^{(N)}) \right\| \\
&\geq \left\| \sum_i t_i Y_i^{(N)} \right\| - \left\| \sum_i t_i (Y_i^{(N)} - X_i^{(N)}) \right\|
\end{aligned}$$

by the triangle property of the norm, and since each column of $\{ \{ k_{ij}^{(N)} \} \}_n$ is infinitely repeated. Therefore $\|P^{(N)}x\| = \|x^{(0,N)}\| \leq 2\|x\| + 2\|\sum_i t_i (Y_i^{(N)} - X_i^{(N)})\|$. Taking the limit as $N \rightarrow \infty$ ($\nu \rightarrow \infty$) in this inequality, we see that $\|Px\| \leq 2\|x\|$ (since $\|P^{(N)}x - Px\| \geq \|P^{(N)}x\| - \|Px\| \rightarrow 0$, $\|P^{(N)}x\| \rightarrow \|Px\|$).

The operation P , defined for each $x \in B_n$ or B_f , is clearly a projection of norm 2 of B_n or B_f on (c_0) , and our theorem is proved. (In case of B_f , as in Theorem 1 P has a unique extension to $\bar{B}_f = B_\infty = W$.)

The foregoing discussion has produced instances of the existence of projections defined on subspaces of (m) to (c_0) , culminating in Theorem 5. Using the result of Phillips mentioned above, we now show that there is no projection defined on (m) to (c_0) .⁹ This result is perhaps rather surprising, since in all of our previous instances, not only did projections exist, but also there were projections of norm exactly 2. (One might expect to be able to find easily a succession of larger and larger subspaces containing (c_0) , for which the greatest lower bounds of the norms of projections would increase unboundedly.) Theorem 5 shows that the nonexistence of a projection on (m) to (c_0) is due in an essential way to the *inseparability* of (m) .

Phillips' result is that there is no projection on (m) to (c) . It follows easily from this that there is no projection on (m) to (c_0) , in view of the following simple theorem.

THEOREM 6. *Suppose that l_1, l_2 are any pair of complementary subspaces in a Banach space L , and that l'_2, l''_2 are any pair of complementary subspaces in l_2 . Let $L_1 = (l_1 + l'_2)$. Then L_1 is closed in L , and L_1 and $L_2 = l''_2$ are complementary subspaces in L .*

PROOF. Let P be the projection of L on l_1 according to l_2 . Suppose that $\{x_n\}$ is any sequence of L_1 , such that $x_n \rightarrow x$ in L , and let $x_n = x_{n1} + x_{n2}'$, where $x_{n1} \in l_1$, $x_{n2}' \in l'_2$. Then $Px_n = x_{n1} \rightarrow Px = x_1 \in l_1$, and $(I - P)x_n = x_{n2}' \rightarrow (I - P)x = x_2' \in l'_2$, by the continuity of P and of $I - P$. Therefore $x_n = x_{n1} + x_{n2}' \rightarrow x_1 + x_2' = x$, so that $x \in L_1$, and L_1 is closed. Moreover, l_1 and l''_2 are complementary subspaces in L_1 .

⁹ "Projection of (m) on (c_0) " of course means the same as "projection on (m) to (c_0) ."

By hypothesis, for any $x \in L$, $x = x_1 + x_2 = x_1 + x'_2 + x''_2$, where $x_1 \in l_1$, $x'_2 \in l'_2$, $x''_2 \in l''_2$, and this decomposition is unique. Therefore the decomposition $x = (x_1 + x'_2) + x''_2$ according to L_1 and L_2 also is unique. This verifies Theorem 6.

If there were a projection of (m) on (c_0) , there would be a complementary subspace l_2 to $l_1 = (c_0)$ in (m) . The element $X_1 = (1, 1, \dots) \in (m)$ would have a decomposition $X_1 = x^{(0)} + X$, $x^{(0)} \in (c_0)$, $X \in l_2$. Consider the one-dimensional subspace consisting of all elements of the form ιX . By the Hahn-Banach theorem, there exists a projection of norm 1 on any one-dimensional subspace. Thus if $l'_2 = \{\iota X\}$, there exists a complementary subspace l''_2 in l_2 . The space $L_1 = (l_1 + l'_2)$ of Theorem 6 is (c) , and by Theorem 6 and the lemma of Murray already mentioned, there would exist a projection of (m) on (c) , contradicting the result of Phillips. Thus there can be no projection of (m) on (c_0) .

By an argument identical with that used by Phillips to prove the nonexistence of a projection of (m) on (c) , it may also be shown directly that there is no projection of (m) on (c_0) . Theorem 6 will be required, however, for another remark to be made below. (It should perhaps be mentioned that (c) is isomorphic to (c_0) .¹⁰)

By Theorem 5 and since there is no projection of (m) on (c_0) , there is no projection of (m) on any separable subspace $W \supset (c_0)$. (For if Q were a projection of (m) on W , P a projection of W on (c_0) , then PQ would be a projection of (m) on (c_0) .) The following question now arises: Does there exist a closed linear subspace L of (m) , $L \supset (c_0)$, such that there is no projection of (m) on L , and no projection of L on (c_0) ? Of course by Theorem 5, such a space L cannot be separable.

An extension theorem found in the paper by Phillips which has been cited¹¹ is as follows. Let X be any linear subspace of a Banach space Z , and let U be any linear transformation on X to (m) . If $y = \{y_i\} = Ux$, the functionals $y_i(x)$ are obviously linear, and by the Hahn-Banach theorem they may be extended to Z with preservation of the norm. Let $\bar{z}_i(z)$ denote the extension of $y_i(x)$. Then $U_1 z = \{\bar{z}_i(z)\}$ is a linear transformation defined on Z with range (m) , which coincides with U on X , and $|U_1| = |U|$. As a consequence of this, if X is isomorphic to any subspace (B) of (m) , and if there exists any Banach space $W \supset X$ such that there is no projection of W on X , then there is no projection of (m) on (B) . For if U is the isomorphism, U_1 its extension on W , and if Q were the projection of (m) on (B) , then $U^{-1}QU_1$ would be a projection of W on X .

¹⁰ Banach, op. cit., p. 181.

¹¹ P. 538, Theorem 7.1, Corollary 7.2.

It follows immediately by the previous paragraph that not only is there no projection of (m) on (c_0) in the usual orientation of (c_0) with respect to (m) , but also that there is no projection of (m) on (c_0) in any imbedding whatever of (c_0) in (m) . (Let W be (m) , let X be (c_0) in the usual imbedding, let (B) be (c_0) in an arbitrary imbedding in (m) ; and take U to be the identity on (c_0) to (c_0) .) Also, since as has been shown by Phillips¹² there exists no projection of (C_1) (the space of functions having only discontinuities of the first kind) on (C) , it follows that there is no projection of (m) on (C) in the imbedding of Theorem 4, or in any other imbedding.

Another interesting consequence of Phillips' extension theorem is that for every Banach space $Z \supset (m)$, there is a projection of norm 1 on Z to (m) . (Extend to Z the identity transformation of $(m) \subset Z$ into (m) .) Also, consider the finite dimensional spaces $l_{\infty, n}$ of sequences $x = (x_1, \dots, x_n)$, with norm $\|x\| = \max_i |x_i|$. In a similar way, it follows that there is a projection of norm 1 on any Banach space $Z \supset l_{\infty, n}$ to $l_{\infty, n}$. There is probably some connection between this and the result of Theorem 5, since $(c_0) \supset l_{\infty, n}$ for every n . A question which arises here is whether the most general separable Banach space having (c_0) as a subspace is contained in (m) in the same relationship to (c_0) as in Theorem 5. Anyway, if B is any separable Banach space having a subspace equivalent to (c_0) , there is a projection of norm 2 of B on (c_0) ; for let U be the identity on (c_0) in B to (c_0) in (m) , extend U to be U_1 on B to (m) by Phillips' theorem, and let the proper range of U_1 be W' . By continuity of U_1 , W' is separable. Let P' be a projection of norm 2 on W' to (c_0) , as given by Theorem 5. Then $U^{-1}P'U_1$ is the required projection of norm 2 on B to (c_0) .

Any separable Banach space Y is equivalent to a subspace (B) of (m) .¹³ The question of whether there is an infinite dimensional, separable Banach subspace (B) in (m) , such that there is a projection on (m) to (B) , is equivalent to the question of whether there is a separable Banach space Y , such that for every Banach space $W \supset Y$, there is a projection on W to Y . A further question is whether such a subspace is necessarily reflexive. ((c_0) is not reflexive.)

Let Y be any separable Banach space; imbed Y in (C) and (C) in (m) . If there is no projection on (C) to Y , then there is no projection on (m) to Y in any imbedding. If there is a projection on (C) to Y , then using Theorem 6 and the facts that any two complementary subspaces of the same closed linear subspace are isomorphic, and that

¹² Loc. cit., p. 539.

¹³ A direct imbedding is given by Phillips, loc. cit., p. 524.

there is no projection on (m) to (C) , it may be shown that at least either there is no projection on (m) to Y , or else there is no projection on (m) to the complementary subspace of Y in (C) . (An illustration of the case where there is no projection on (m) to the complementary subspace in (C) is provided by the case of a finite dimensional $YC(C)$.)

In a paper in preparation on the extension of linear transformations, the writer intends to discuss the questions indicated above, and related questions.

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SEQUENCES OF STIELTJES INTEGRALS¹

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Statement of results. Sequences of Riemann-Stieltjes integrals² have as yet been little studied, only the following fundamental results being known.

THEOREM A (Helly [2]). *Let $g_n(x)$ ($n=1, 2, \dots$) be an infinite sequence of real functions defined in the finite closed interval $I=(a, b)$ which satisfy the following two conditions:*

- (1) *Total variation of g_n in $I \equiv V_I(g_n) \leq M$, M a fixed constant,*
- (2) *$g_n \rightarrow g$ on I , $n \rightarrow \infty$;*

then for any function $f(x)$ continuous in I , we have³

$$(3) \quad \int f dg_n \rightarrow \int f dg.$$

THEOREM B (Shohat [3]). *Let $\{g_n\}$ be a sequence of functions monotonic and uniformly bounded in I and such that*

- (4) *$g_n \rightarrow g$ on E , E a set dense on I and including the end points a, b of I , where g is a monotonic function (all the functions g_n, g monotonic in the same sense); then we have (3) for any function $f(x)$ for which*

¹ Presented to the Society, January 1, 1941.

² A discussion of such integrals with references is to be found in [1]. (Numbers in brackets refer to the bibliography.)

³ When the limits of integration are omitted, it is to be understood that they are the end points a, b of I .

$$(5) \quad \int f dg_n \text{ exist,} \quad n = 1, 2, \dots,$$

$$(6) \quad \int f dg \text{ exists.}$$

Theorem A was rediscovered by Bray [4]⁴ and a condition weaker than (4), which is both necessary and sufficient, was found by Hildebrandt [5].⁵

The present paper is devoted to a preliminary study of conditions governing convergence of the sequences in question when neither continuity of f nor monotonicity of g_n is assumed. The goal, not attained here, is to find necessary and sufficient conditions to be imposed on the sequence $\{g_n\}$, where the functions g_n belong to the class V of functions of bounded variation in I , in order to insure the validity of (3) for every bounded f for which (5), or (5) and (6) hold true. Two necessary conditions can be given at once. Condition (1) is necessary since it is already necessary if we require that f belong only to the class of functions continuous in I [5]. It is interesting to note, however, that condition (1) is not necessary if we require that f belong only to V , as is seen by the following theorem:

THEOREM 1. *If we have (2), then in order that we have (3) for every function f of V for which (5) and (6) are true, it is sufficient that the sequence $\{g_n\}$ be uniformly bounded.*

The other necessary condition is obtained by taking for f functions which are constant everywhere in I except for a single point. It is easily found that the set C of points of convergence of g_n to g must include all the common continuity points of g_n , g as well as the end points a , b of I . Without going into the question of necessity⁶ we shall assume that C contains all the continuity points of g (whether they are also continuity points of all g_n or not), that is, that

$$(7) \quad \begin{array}{l} g_n \rightarrow g \text{ on } C, \quad g \text{ in } V, \\ C \text{ consisting of the continuity points of } g \text{ and of } a \text{ and } b. \end{array}$$

⁴ In Bray's formulation of the theorem condition (2) is replaced by (4); but it is then necessary to specify that g is a function of bounded variation; and by the lemma given below, it is seen that the two formulations are completely equivalent.

⁵ The following additional reference was brought to the author's attention by the referee: Hildebrandt, *Stieltjes integration of the Riemann type*, American Mathematical Monthly, vol. 45 (1938), pp. 265-278, Theorems 2.41; 5.41.

⁶ It is to be noted that in Theorem B as well as in its generalization, Theorem 3, conditions (4) and (7) are equivalent, as can be readily proved.

The question arises whether condition (7) represents in the problem under consideration an essential weakening of condition (2). That this is not the case is shown by the following lemma.

LEMMA. *Under the conditions (1), (5), and (6), if (3) is implied by (2), it will also be implied by (7). If we are considering only a certain f satisfying our conditions, then C need consist of only those continuity points of g which are also discontinuity points of f (in addition to being dense on I and containing the points a and b).*

It is thus no loss of generality to assume condition (2) rather than (7) in the following theorems. When this is the only condition imposed on the sequence $\{g_n\}$, only the following result (besides that given in Theorem 1) has so far been obtained.

THEOREM 2. *If f has at most singularities of the first kind, then (1), (2), (5), and (6) imply (3).*

A direct generalization of Theorem B is given in the following theorem.

THEOREM 3. *If in addition to (2) we assume that*

$$(8) \quad V_I(g_n) \rightarrow V_I(g), \quad g_n, g \text{ in } V,$$

then (5) and (6) imply (3). (Condition (1) is of course implied by (8).)

The next theorem gives a result of wider scope.

THEOREM 4. *To insure the validity of (3) it is sufficient to assume in addition to (1), (2), and (5) that⁷*

$$(9) \quad \text{for every limit function } v(x) \text{ of the sequence } \{V_a^x(g_n)\} \int f dv \text{ exists.}$$

In some applications the interval of integration is infinite and it is therefore of interest to consider that case. This is treated here only briefly, the following results being immediate.⁸

THEOREM 5. *For $I = (a, b) = (a, \infty)$, if we assume (2), (5), and*

$$(10) \quad \int_a^c f dg_n \rightarrow \int_a^c f dg \text{ for every finite } c > a; \quad \int_a^c f dg \text{ exists,}$$

⁷ The existence of at least one limit function of the sequence given in (9) is insured by Helly's theorem of choice [2]. Note also that (6) is implied by (9), as can be seen, for example, by using the integrability condition on p. 542 [1] along with Theorem 2 in [7].

⁸ In order to simplify the formulas, we take only one of the limits as infinite.

then a necessary and sufficient condition for the validity of (6) and (3) is that we have either one of the following two conditions (ϵ being an arbitrarily assigned positive number):

$$(11a) \quad \left| \int_c^\infty f dg_n \right| < \epsilon, \quad c \geq c_0, n \geq N(\epsilon, c),$$

$$(11b) \quad \left| \int_a^c f d(g_n - g) \right| < \epsilon, \quad n \geq n_0, c \geq C(\epsilon, n).$$

NOTE 1. Condition (10) of the theorem can of course be replaced by the appropriate conditions in the preceding theorems which insure the validity of (10); thus if f is continuous in I , we can replace (10) by the requirement:

$$(12) \quad V_a^c(g_n) \leq M(c).$$

NOTE 2. If f is bounded in I , then (11a) can be replaced by the relation $V_c^\infty(g_n) < \epsilon$, $c \geq c_0$, $n \geq N(\epsilon, c)$.

NOTE 3. If f is bounded in I and g is of bounded variation for sufficiently large x , then we need assume only the validity of a relation like (4), and (11) can be replaced by the following corresponding relations:

$$\begin{aligned} \left| \int_{a_k}^\infty f dg_n \right| &< \epsilon, \quad k \geq k_0, n \geq N(\epsilon, k), \\ \left| \int_a^{a_k} f d(g_n - g) \right| &< \epsilon, \quad n \geq n_0, k \geq K(\epsilon, n), \end{aligned}$$

where $\{a_k\}$ is some sequence in E converging to ∞ .

NOTE 4. If we have (12) and if

$$\lim_{c \rightarrow \infty} \overline{B} \{ |f(x)| \} \cdot M(c) = 0,$$

then we certainly have (11a). (The above relation will hold for instance when $\lim_{x \rightarrow \infty} f(x) = 0$ and $M(c)$ is independent of c .)

By the use of Theorem 5 we can give a direct proof of the existence of a solution of the moments-problem of Stieltjes-Hamburger.⁹ This proof as well as other applications of the above theorems will be given elsewhere.

Proof of the lemma. The proof will be based on the following auxiliary lemma.

⁹ The proof given in the literature is indirect, involving complex function theory. A discussion of this problem and references are to be found in [6].

LEMMA. *Let g and h be two functions of V which coincide in value on a set D , and let f be a function bounded in I . Then the existence of the integral of f with respect to one of these functions, say g , will imply its existence with respect to the other function h , and the equality of the values of the two integrals, provided D is dense in I and includes the end points of I as well as all those discontinuity points of f at which h has an external saltus.*

PROOF. That $\int f dg = \int f dh$ when both integrals exist follows directly by the definition of Stieltjes integrals and the fact that D is dense in I and includes the end points of I . To prove that $\int f dh$ exists it is clearly sufficient to prove that $\int f dk$ exists, where $k = g - h$; and for that it is only necessary to show that the variation of k over the set of discontinuities of f is zero [1]. This can be readily shown when we note that by our assumption $k(x-0) = k(x+0) = 0$, and that therefore, if we denote the discontinuity points of k by u_i ($i = 1, 2, \dots$), the total variation of k over the set obtained by excluding from I arbitrarily small intervals about the points $u_1, \dots, u_m$, can be made as small as we please provided m is taken sufficiently great; and that further by our assumptions, f is continuous at each u_i .

To prove the lemma, we note that by (1) the sequence $\{\int f dg_n\} \equiv \{J_n\}$ is bounded and it therefore contains at least one convergent subsequence. Let $\{J_{n'}\}$ be any such convergent subsequence and denote its limit by J ; let $\{g_{n''}\}$ be a convergent subsequence of the function-sequence $\{g_{n'}\}$ (existing by Helly's theorem of choice [2]), and denote its limit function by g^* . Now it can easily be proved that by the condition given in the lemma, no limit function of $\{g_n\}$ can have an external saltus at a discontinuity point of f . We can apply therefore the above auxiliary lemma and conclude that $\int f dg^*$ exists and equals $\int f dg$. Hence by our assumption, it follows that $\int f dg_{n''} \rightarrow \int f dg$, and consequently also $J = \int f dg$. But $\{J_{n'}\}$ was an arbitrary convergent subsequence of $\{J_n\}$; hence $\{J_n\}$ has the unique limit J , that is, we have (3).

Proof of Theorem 1. It is clearly sufficient to consider f monotonic. By the formula of integration by parts for Stieltjes integrals [1], it is seen that the theorem will follow if we show that

$$(*) \quad \int g_n df \rightarrow \int g df.$$

To this end, we employ Lebesgue's theorem on the convergence of sequences of Lebesgue integrals with boundedly convergent integrands.

Putting $A=f(a)$, $B=f(b)$, $G_n(y) \equiv g_n(f^{-1}(y))$, $G(y) \equiv g(f^{-1}(y))$, we have [1]:

$$\int g_n df = L \int_A^B G_n(y) dy, \quad \int g df = L \int_A^B G(y) dy;$$

and as by our assumptions, all the conditions of Lebesgue's theorem are satisfied, it follows that $\int G_n(y) dy \rightarrow \int G(y) dy$, and that we have (*).

Proof of Theorem 2. This theorem follows from the following three lemmas (Lemma 2 is more general than the present use of it requires).

LEMMA 1. *Theorem 2 is true if f is in V .*

PROOF. Let $f=f_c-f_d$ be the decomposition of f into its continuous part f_c and its discontinuous part f_d [1]. By Theorem A, $\int f_c dg_n \rightarrow \int f_c dg$, and it remains to be shown that we also have the relation $\int f_d dg_n \rightarrow \int f_d dg$, or the relation corresponding to (*) above. Now $\int g_n df_d = \sum_m d_m g_n(u_m)$, where $d_m = f(u_m+0) - f(u_m-0)$, and as the series $\sum d_m g_n(u_m)$ converges uniformly with respect to n , being dominated by

$$\overline{B}\{|g_n|\} \cdot \sum |d_m|$$

which is finite by our assumptions, it follows that

$$\lim_{n \rightarrow \infty} \int g_n df_d = \sum_m d_m g(u_m) = \int g df_d.$$

LEMMA 2. *If the sequence of bounded functions f_m converges uniformly in I to the bounded function f , and if we have (5), (6), and (3) with f replaced by f_m ($m=1, 2, \dots$) then we have (5), (6), and (3), provided we assume (1) and the finiteness of $V_I(g)$.*

PROOF. We show first that the existence of $\int f_m dg$ and the uniform convergence of f_m to f in I implies the relation

$$(*) \quad \int f dg \text{ exists,} \quad \int f_m dg \rightarrow \int f dg,$$

g being assumed to belong to V . Let

$$s_{mk} = \sum_{i=1}^{r_k} f_m(u_i) [g(x_i) - g(x_{i-1})]$$

be the k th Riemann sum of a sequence whose limit for $k \rightarrow \infty$ is $\int f_m dg$, and let s_k represent the corresponding sum for f . As

$$|s_{mk} - s_k| \leq \overline{B} \left\{ |f_m - f| \right\} \cdot V_I(g),$$

we see that s_{nk} approaches its limit s_k uniformly with respect to k , and that therefore $\lim_{k \rightarrow \infty} s_k$ and $\lim_{n \rightarrow \infty} \int f_n dg$ exist and are equal. Relation (*) now follows from the fact that the first limit is independent of the choice of the division points x_i and the between-points u_i in the Riemann sums.

Now by (1),

$$\left| \int f dg_n - \int f_n dg_n \right| \leq \overline{B}_{x \text{ in } I} \{ |f - f_n| \} \cdot M,$$

that is, $\int f_n dg_n$ converges to $\int f dg_n$ uniformly with respect to n . Hence

$$\begin{aligned} \lim_n \int f dg_n &= \lim_n \lim_m \int f_m dg_n = \lim_m \lim_n \int f_m dg_n \\ &= \lim_m \int f_m dg = \int f dg, \end{aligned} \quad \text{by (*).}$$

LEMMA 3. *If f is a function having at most singularities of the first kind in I , then there exists a sequence $\{f_m\}$ converging to f uniformly in I , where each f_m is in V and is continuous at the continuity points of f .*

PROOF. We have to show that we can assign to every given positive and arbitrarily small number ϵ , a function f_ϵ , which is in V , is continuous at every continuity point of f , and approximates f within ϵ . By assumption, $f(x-0)$ and $f(x+0)$ exist, and we can therefore assign to every x of the semi-open interval $(a, b]$,¹⁰ a left-side interval (x', x) , and to every x of $[a, b)$, a right-side interval (x, x'') , in each of which the oscillation of f is less than $\epsilon/4$. By the Heine-Borel covering theorem, there exists a finite set of points x_i in I such that the sum of the corresponding intervals $(x'_i, x''_i) \equiv I_i$ (including the intervals $[a, a'')$ and $(b', b]$) covers I , and we may assume without loss of generality that the overlapping parts (x'_{i+1}, x''_i) of consecutive intervals I_i (necessarily nonvacuous, as the I_i are open) lie inside (x_i, x_{i+1}) . It can now be seen without difficulty that we can take as the required function f_ϵ , a function defined as follows: It has the values $f(x_i)$, $f(x_i-0)$, and $f(x_i+0)$, respectively for $x=x_i$, for x in (x'_{i-1}, x_i) , and for x in (x_i, x_{i+1}) ; and it is linear in the remaining intervals.

It is apparent how Theorem 2 can be derived by the use of the above three lemmas. We need only observe that the functions f_m of Lemma 3 (with f of Theorem 2) satisfy the relations (5) and (6) (with $f=f_m$) because they are in V , and have no common singularity

¹⁰ The *parenthesis* will represent the *open*, and the *bracket*, the *closed* end of an interval.

points with the functions g_n, g (by their definition, in view of (5) and (6)). It is in fact known that two functions of bounded variation which have no common discontinuity points, are integrable with respect to each other.¹¹

Proof of Theorem 3. Conditions (2) and (8) are equivalent [7] to (2) and the relation

$$(8') \quad V_a^x(g_n) \rightarrow V_a^x(g), \quad n \rightarrow \infty; x \text{ in } I.$$

Hence, denoting by $g_n = p_n - q_n$, $g = p - q$ the Jordan decompositions of the functions g_n, g , we readily see that all the conditions of Theorem B apply to the two sequences $\{p_n\}, \{q_n\}$ ($\int f d p_n, \int f d q_n$ exist because $\int f d g_n$ exist [1]), and that therefore: $\int f d p_n \rightarrow \int f d p$, $\int f d q_n \rightarrow \int f d q$, and hence (3) is satisfied.

The theorem can also be proved directly as follows:¹² Let

$$s_{nk} = \sum_{i=1}^{r_k} f(u_{i,k}) [g_n(x_{i,k}) - g_n(x_{i-1,k})]$$

represent the k th Riemann sum of a sequence of such sums which converges to $\int f d g_n \equiv J_n$ when $k \rightarrow \infty$, then:

$$(*) \quad |J_n - s_{nk}| \leq \sum_{i=1}^{r_k} \text{Osc}_i f(x) V_i(g_n),$$

the subscript i referring to the i th subinterval $(x_{i-1,k}, x_{i,k})$. By (6), given a positive number h , there exists a number $K = K(h)$ such that $O_k \equiv \sum_i \text{Osc}_i f(x) V_i(g) < h$, $k \geq K(h)$. Now let ϵ be a preassigned positive number. Fixing first $k' \geq K(\epsilon/2)$, we can then choose, in view of (8'), a number $N = N(\epsilon, k')$ so that for all $n \geq N$, we have

$$|V_a^{x_i, k'}(g_n) - V_a^{x_i, k'}(g)| < \frac{\epsilon}{4r_{k'} U} \equiv \epsilon', \quad U = \text{Osc}_I f; i = 0, 1, \dots, r_{k'}.$$

Then

$$\sum_{i=1}^{r_{k'}} \text{Osc}_i f(x) V_i(g_n) < O_{k'} + 2\epsilon' r_{k'} U < \epsilon/2 + \epsilon/2 = \epsilon,$$

$$k' \geq K(\epsilon/2) = k_*; n \geq N(\epsilon, k').$$

¹¹ We note, incidentally, that by Lemma 3 and the first part of the proof of Lemma 2, it follows that a bounded function which has at most singularities of the first kind is integrable with respect to every function of V with which it has no common singularities.

¹² This proof, which is different from that given in [3] for Theorem B, was found by the author in the winter of 1939, when most of the results given in this paper were developed.

Combining this with (*), and remembering that ϵ was arbitrarily assigned, we finally conclude that $\lim_{n \rightarrow \infty} J_n = \lim_k \lim_n s_{nk} = \iint f dg$, that is, that we have (3).

Proof of Theorem 4. Let us suppose first that the limit function $v(x)$ occurring in (9) is unique. Then if $g_n = p_n - q_n$ represent the Jordan decompositions of the functions g_n , we find:

$$2p_n(x) \rightarrow v(x) + g(x) + g(a) \equiv p^*(x), \quad 2q_n \rightarrow v(x) - g(x) + g(a) \equiv q^*(x).$$

Since by (6) and (9), the integrals of f with respect to p^* and q^* exist, we can now proceed exactly as in the first proof of Theorem 3.

In the general case, let $\{J_n\}$ be an arbitrary convergent subsequence of $\{J_n\} \equiv \{\iint f dg_n\}$. By Helly's theorem of choice, in view of (1) and (2), there exists a convergent subsequence $\{v_{n'}(x)\}$ of the sequence $\{v_n(x)\} \equiv \{V_a^*(g_n)\}$, and it follows by the preceding discussion that $J_{n'} \rightarrow \iint f dg$. By a familiar reasoning it then follows that we also have the relation $J_n \rightarrow \iint f dg$, that is, (3).

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HEAT CONDUCTION IN AN INFINITE COMPOSITE SOLID¹

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1. **Introduction.** The problem of one-dimensional heat conduction in finite or semi-infinite composite solids has been investigated extensively. The doubly-infinite case, however, seems to have been treated only for special initial temperature distribution functions.²

The purpose of the present paper is to treat the general case. The Laplace transformation is used formally in §2 to discover the solution, which is then rigorously established in §3. In §4, finally, a uniqueness-theorem is proved, under more restrictive conditions on the initial distribution function.

Lebesgue integrals are used throughout.

2. **The formal solution.** Consider two plane-boundary semi-infinite homogeneous solids, composed of different materials, placed in perfect thermal contact. If the conduction of heat takes place in only one dimension, perpendicular to the interface, the temperature $U(x, t)$ satisfies the following differential system:

$$(1) \quad \frac{\partial U}{\partial t} = a_\nu \frac{\partial^2 U}{\partial x^2}; \quad t > 0; x < 0, \nu = 1; x > 0, \nu = 2;$$

$$(2) \quad \lim_{t \rightarrow 0} U(x, t) = f(x); \quad x \neq 0;$$

$$(3) \quad \lim_{x \rightarrow -0} U(x, t) = \lim_{x \rightarrow +0} U(x, t); \quad t > 0;$$

$$(4) \quad \lim_{x \rightarrow -0} k_1 \frac{\partial U}{\partial x} = \lim_{x \rightarrow +0} k_2 \frac{\partial U}{\partial x}; \quad t > 0;$$

where x is the perpendicular distance from the interface, t is time, a_ν and k_ν are the thermal diffusivities and conductivities, respectively, of the two materials and are positive constants, and $f(x)$ is a known function, defined for all real x except $x=0$, whose properties will be specified later.

Denoting the common limit in equation (3) by $\Phi(t)$, the solution can be written in the well known form³

¹ Presented to the Society, April 5, 1941.

² Cf. Riemann-Weber, *Die partiellen Differential-Gleichungen der mathematischen Physik*, 5th edition, 1912, vol. 2, p. 98.

³ Cf. H. S. Carslaw, *The Mathematical Theory of the Conduction of Heat in Solids*, 2d edition, London, 1921, §§18, 23.

$$(5)^4 \quad U(x, t) = V_2(x, t) + \frac{|x|}{2(\pi a_1 t)^{1/2}} \int_0^t \Phi(t - \xi) \xi^{-3/2} \exp \left[-\frac{x^2}{4a_1 \xi} \right] d\xi,$$

where

$$V_1(x, t) = \frac{1}{2(\pi a_1 t)^{1/2}} \int_{-\infty}^0 f(\xi) \left\{ \exp \left[-\frac{(x - \xi)^2}{4a_1 t} \right] - \exp \left[-\frac{(x + \xi)^2}{4a_1 t} \right] \right\} d\xi,$$

and $V_2(x, t)$ is obtained from $V_1(x, t)$ on replacing a_1 by a_2 and integrating from 0 to $+\infty$.

Now apply the Laplace transformation to equations (4) and (5), denoting the transforms of U , V , and so on, by u , v , and so on, respectively:

$$(4') \quad \lim_{s \rightarrow -0} k_1 \frac{\partial u(x, s)}{\partial x} = \lim_{s \rightarrow +0} k_2 \frac{\partial u(x, s)}{\partial x},$$

$$(5')^5 \quad u(x, s) = \phi(s) \exp \left[-|x| (s/a_1)^{1/2} \right] + v_2(x, s).$$

The unknown function $\phi(s)$ can be eliminated between equations (4') and (5'), and the solution is then obtained by applying the inverse Laplace transformation to $u(x, s)$:

$$\begin{aligned} U(x, t) = V_1(x, t) + \frac{k_1}{A a_1 (\pi t)^{1/2}} \int_{-\infty}^0 f(\xi) \exp \left[-\frac{(x + \xi)^2}{4a_1 t} \right] d\xi \\ + \frac{k_2}{A a_2 (\pi t)^{1/2}} \int_0^{\infty} f(\xi) \exp \left[-\frac{(x a_2^{1/2} - \xi a_1^{1/2})^2}{4a_1 a_2 t} \right] d\xi; \end{aligned}$$

$x < 0, t > 0;$

$$\begin{aligned} (6) \quad U(x, t) = V_2(x, t) + \frac{k_2}{A a_2 (\pi t)^{1/2}} \int_0^{\infty} f(\xi) \exp \left[-\frac{(x + \xi)^2}{4a_2 t} \right] d\xi \\ + \frac{k_1}{A a_1 (\pi t)^{1/2}} \int_{-\infty}^0 f(\xi) \exp \left[-\frac{(x a_1^{1/2} - \xi a_2^{1/2})^2}{4a_1 a_2 t} \right] d\xi; \end{aligned}$$

$x > 0, t > 0,$

where $A = (k_1(a_2)^{1/2} + k_2(a_1)^{1/2})/(a_1 a_2)^{1/2}$.

⁴ Throughout, $x < 0$ when $\nu = 1$, $x > 0$ when $\nu = 2$.

⁵ The first term in the right member is obtained by the "Faltung" rule. For it and the specific transformations used, cf. G. Doetsch, *Theorie und Anwendung der Laplace Transformation*, Berlin, 1937, particularly the table of transformations in Appendix 2.

3. The solution established. We have the following theorem.

THEOREM 1. *If $f(x)$ is Lebesgue integrable over any finite interval, and if there exist two positive numbers B, b , such that $|f(x)| < B \exp |bx|$, then the function $U(x, t)$ defined by equations (6) is a solution of the boundary value problem (1), (3), (4). The initial condition (2) is satisfied uniformly in x in any finite interval $0 < x_1 \leq |x| \leq x_2$ in which $f(x)$ is continuous.*

In the classical proofs of similar theorems^a $f(x)$ is always a bounded function. In adapting such methods of proof to the present situation it is only necessary to investigate the effect of the unboundedness of $f(x)$. A complete proof is given here for only one typical integral in each case. The right members of (6) obviously converge.

(a) *The differential equation.* If $U(x, t)$ is differentiated with respect to t under the integral sign, one of the resulting integrals is of the form

$$J(x, t) \equiv t^{-5/2} \int_0^\infty (x \pm \xi)^2 f(\xi) \exp \left[-\frac{(x \pm \xi)^2}{4t} \right] d\xi.$$

We prove that, if $x > 0$, $0 < t_1 \leq t \leq t_2$, $J(x, t)$ converges uniformly in t . From the hypotheses on $f(x)$,

$$|J(x, t)| \leq B t_1^{-5/2} \int_0^\infty (x \pm \xi)^2 e^{b\xi} \exp \left[-\frac{(x \pm \xi)^2}{4t_2} \right] d\xi.$$

Since the right member converges, and does not contain t , $J(x, t)$ converges uniformly in t .

(b) *The initial condition.* The solutions (6) are composed of two essentially distinct types of integrals, according as x and ξ have the same or opposite signs. We first prove that, in the first case, the integral vanishes with t .

Consider

$$J(x, t) \equiv \frac{1}{2(\pi t)^{1/2}} \int_0^\infty f(\xi) \exp \left[-\frac{(x + \xi)^2}{4t} \right] d\xi, \quad x > 0.$$

Make the change of variable $\xi = -x + 2\zeta t^{1/2}$, then:

$$J(x, t) = \pi^{-1/2} \int_{x/2t^{1/2}}^\infty f(-x + 2\zeta t^{1/2}) e^{-\zeta^2} d\zeta.$$

From the hypotheses on $f(x)$,

^a Cf. Carslaw, loc. cit., p. 31; Goursat, *Cours d'Analyse Mathématique*, 4th edition, Paris, 1927, vol. 3, chap. 29.

$$|J(x, t)| \leq B e^{-bx\pi^{-1/2}} \int_{x/2t^{1/2}}^{\infty} \exp[-\xi(\xi - 2bt^{1/2})] d\xi.$$

If $x \geq x_1 > 0$, and $t < x_1/8b$, then

$$|J(x, t)| \leq \frac{4B}{x_1} (t/\pi)^{1/2} \exp[-bx_1 - x_1^2/8t].$$

Hence $J(x, t)$ approaches zero with t , uniformly in x in any semi-infinite interval $0 < x_1 \leq x$.

An inspection of (6) now shows that, in order to prove that the initial condition (2) is satisfied it is sufficient to prove that

$$\lim_{t \rightarrow 0} [J_1(x, t) - f(x)] = 0$$

where

$$J_1(x, t) = \frac{1}{2(\pi t)^{1/2}} \int_0^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi, \quad x > 0,$$

the approach to the limit being uniform in x in any interval $0 < x_1 \leq x \leq x_2$. Given any $\epsilon > 0$, we can choose a $\delta > 0$, independent of x and t , such that $|f(\xi) - f(x)| < \epsilon$ if $|x - \xi| < \delta$, $\delta < x_1$. Having chosen δ , we can write

$$(7) \quad |J_1(x, t) - f(x)| \leq \int_0^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty} + \frac{|f(x)|}{\pi^{1/2}} \int_{x/2t^{1/2}}^{\infty} e^{-\xi^2} d\xi,$$

where in each of the first three integrals the integrand is

$$\frac{1}{2(\pi t)^{1/2}} |f(\xi) - f(x)| \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi.$$

Denote the right member of (7) by $I_1 + I_2 + I_3 + I_4$. The unboundedness of $f(x)$ is important in I_1 , I_3 , and I_4 . I_2 is easily seen to be less than ϵ . In I_1 make the change of variable $\xi = x - 2\zeta t^{1/2}$. Then

$$I_1 + I_4 \leq 3B e^{bx_2\pi^{-1/2}} \int_{\delta/2t^{1/2}}^{\infty} e^{-\zeta^2} d\zeta,$$

which obviously approaches zero with t , uniformly in x .

From the hypotheses on $f(x)$,

$$I_3 \leq \frac{B}{(\pi t)^{1/2}} \int_{x+\delta}^{\infty} \exp\left[b\xi - \frac{(x-\xi)^2}{4t}\right] d\xi.$$

Make the change of variable $\xi = x + 8t\zeta/\delta$; if $t < \delta/8b$,

$$I_3 \leq \frac{8B}{\delta} e^{bx}(t/\pi)^{1/2} \int_{\delta^2/8t}^{\infty} e^{-\zeta} d\zeta,$$

which obviously approaches zero with t , uniformly in x .

(c) *The interface conditions.* Equations (3) and (4) will obviously be satisfied if x can approach zero within the integral signs. The proof is given for one typical case:

$$J(x, t) \equiv t^{-3/2} \int_0^{\infty} (x \pm \xi) f(\xi) \exp \left[-\frac{(x \pm \xi)^2}{4t} \right] d\xi.$$

If $0 < t_1 \leq t \leq t_2$, $x > 0$,

$$(8) \quad \begin{aligned} |J(x, t) - J(0, t)| &\leq xt^{-3/2} \int_0^{\infty} |f(\xi)| \exp \left[-\frac{(x - \xi)^2}{4t} \right] d\xi \\ &+ t^{-3/2} \int_0^{\infty} \xi |f(\xi)| \exp \left[-\frac{\xi^2}{4t} \right] \left| 1 - \exp \left[-\frac{x^2 - 2x\xi}{4t} \right] \right| d\xi. \end{aligned}$$

Denote the right member of (8) by $I_1 + I_2$. In I_1 make the change of variable $\xi = x + 2\zeta t^{1/2}$:

$$I_1 \leq \frac{2Bx}{t} e^{bx} \int_{-x/2t^{1/2}}^{\infty} \exp [-\zeta(\zeta - 2bt^{1/2})] d\zeta.$$

If the range of integration is split at $\zeta = 1 + 2bt^{1/2}$, it is easily seen that

$$I_1 \leq \frac{2Bx}{t_1} e^{bx} \left\{ 1 + \pi^{1/2} \exp [2b(1 + 2bt_2^{1/2})t_2^{1/2}] \right\},$$

which approaches zero with x , uniformly in t .

Since $|1 - \exp y| \leq |y| \exp |y|$, we have

$$I_2 \leq Bxt^{-3/2} \int_0^{\infty} \xi \left| \frac{x - 2\xi}{4t} \right| \exp \left[b\xi - \frac{\xi^2}{4t} + \left| \frac{x^2 - 2x\xi}{4t} \right| \right] d\xi.$$

Split the range of integration at $\xi = 3 + 4bt$; if $x < 1$,

$$\begin{aligned} I_2 &\leq Bxt_1^{-5/2} \exp \left[-\frac{x^2}{4t_1} \right] \left\{ \int_0^{3+4bt_2} \xi(1 + 2\xi) \exp \left[b\xi + \frac{\xi}{2t_1} \right] d\xi \right. \\ &\quad \left. + \int_{3+4bt_1}^{\infty} \xi(1 + 2\xi) \exp [-\xi/4t_2] d\xi \right\}, \end{aligned}$$

which approaches zero with x , uniformly in t .

4. **Uniqueness.** We first state the following definition.

DEFINITION. A function $U(x, t)$ defined for all real x and positive t will be said to satisfy conditions A if:

(A₁) $U(x, t)$ is a continuous function of x in $-\infty < x < \infty$ for any fixed $t > 0$.

(A₂) $U(x, t)$ is a continuous function of t in any finite interval $0 < t_1 \leq t \leq t_2$, the continuity being uniform with respect to x in any finite interval $x_1 \leq x \leq x_2$.

(A₃) $U(x, t)$ is bounded for all real x , all positive t .

(A₄) $\partial U(x, t)/\partial t$ exists and satisfies (A₁) and (A₂).

(A₅) $\partial U(x, t)/\partial x$ and $\partial^2 U(x, t)/\partial x^2$ exist and are continuous functions of x in $-\infty < x < 0$ and $0 < x < \infty$, for any fixed $t > 0$.

(A₆) $|\partial U(x, t)/\partial x| < M/t^{1/2}$, $t > 0$, $x \neq 0$; M is a constant independent of t and x .

(A₇) $U(x, t) \rightarrow 0$ as $|x| \rightarrow \infty$, uniformly in t in any finite interval $0 < t \leq t_1$.

(A₈) $U(x, t) \rightarrow 0$ as $t \rightarrow 0$, uniformly in x in any finite interval $0 < x_1 \leq |x| \leq x_2$.

THEOREM 2. The function $U(x, t) \equiv 0$ is the only function satisfying conditions A that is a solution of the boundary value problem (1)–(4), with $f(x) \equiv 0$.

PROOF. Consider

$$J(x, t) \equiv \frac{1}{2} \int_{-x}^x \frac{k_\nu}{a_\nu} [U(\xi, t)]^2 d\xi, \quad x > 0, t > 0.$$

Assume that, for some $x_1 \neq 0$, $t_1 \neq 0$, $|U(x_1, t_1)| = C > 0$. From condition (A₁), $|U(x, t_1)| > C/2$ in some η -neighborhood of x_1 . Hence $J(x, t_1) > D$, any $x > |x_1| + \eta$, $t_1 > 0$, where

$$D = \frac{k_\nu C^2 \eta}{4a_\nu}; \quad \eta < |x_1|; \quad \nu = 1 \text{ if } x_1 < 0; \quad \nu = 2 \text{ if } x_1 > 0.$$

If M is the bound of condition (A₆), we can choose a fixed $x_2 > |x_1| + \eta$ such that

$$(9) \quad \begin{aligned} |U(x_2, t)| &< \frac{D}{4k_2 M t_1^{1/2}}, & 0 < t \leq t_1, \\ |U(-x_2, t)| &< \frac{D}{4k_1 M t_1^{1/2}}, & 0 < t \leq t_1. \end{aligned}$$

Having chosen x_2 , consider $J(x_2, t)$. From conditions (A₁), (A₂), (A₄),

$J(x_2, t)$ can be differentiated with respect to t under the integral sign; using equation (1),

$$\frac{\partial J(x_2, t)}{\partial t} = \int_{-x_2}^{x_2} k_r U(x, t) \frac{\partial^2 U(x, t)}{\partial x^2} dx.$$

From condition (A₅) this may be integrated by parts; using equations (3) and (4),

$$\frac{\partial J(x_2, t)}{\partial t} = \left[k_r U(x, t) \frac{\partial U(x, t)}{\partial x} \right]_{-x_2}^{x_2} - \int_{-x_2}^{x_2} k_r \left[\frac{\partial U(x, t)}{\partial x} \right]^2 dx, \quad 0 < t \leq t_1.$$

Since the last term is nonpositive, $\partial J/\partial t$ is not greater than the sum of the first two terms. Hence, from (A₆) and equations (9)

$$\frac{\partial J(x_2, t)}{\partial t} < \frac{D}{2(t_1 t)^{1/2}}, \quad 0 < t \leq t_1.$$

From conditions (A₃), (A₈), $J(x_2; +0) = 0$. Therefore

$$J(x_2, t_1) < \frac{D}{2t_1^{1/2}} \int_0^{t_1} t^{-1/2} dt = D.$$

But $J(x_2, t_1) > D$. Thus the assumption that $U(x_1, t_1) \neq 0$ leads to a contradiction if $x_1 \neq 0$, $t_1 \neq 0$, and by the continuity conditions (A₁) and (A₈) the theorem is proved.

We are now in a position to prove a more general uniqueness theorem.

DEFINITION. Let $f(x)$ be defined and continuous for all real x except $x=0$. A function $U(x, t)$ will be said to satisfy conditions B with respect to $f(x)$ if:

(B₁) $U(x, t)$ satisfies (A₁)–(A₈) inclusive.

(B₂) $U(x, t) \rightarrow f(x)$ as $t \rightarrow 0$, uniformly in x in any finite interval $0 < x_1 \leq |x| \leq x_2$.

(B₃) $U(x, t) - f(x) \rightarrow 0$ as $|x| \rightarrow \infty$, uniformly in t in any finite interval $0 < t \leq t_1$.

THEOREM 3. Let $f(x)$ be defined and continuous for all real x except $x=0$. Then the boundary value problem (1)–(4) has at most one solution satisfying conditions B with respect to $f(x)$.

PROOF. If there are two solutions, let $U(x, t)$ be their difference. Then $U(x, t)$ satisfies conditions A and hence vanishes identically.

Finally, it remains to determine conditions on $f(x)$ under which the

solution $U(x, t)$ defined by equations (6) will satisfy conditions B with respect to $f(x)$. Sufficient conditions are given by the following theorem.

THEOREM 4. *Let $f(x)$ be continuous except at $x=0$, and bounded for all real x , and let the following limits exist:*

$$f_1 = \lim_{x \rightarrow -\infty} f(x), \quad f_2 = \lim_{x \rightarrow +\infty} f(x).$$

Then $U(x, t)$ as defined by equations (6) satisfies conditions B with respect to $f(x)$.

PROOF. Conditions (A_1) – (A_5) and (B_2) are easily seen to be satisfied. We give the proof for (A_6) and (B_3) . An inspection of (6) shows that it is sufficient to prove (A_6) for integrals of the form

$$J(x, t) \equiv \frac{1}{2(\pi t)^{1/2}} \int_0^\infty f(\xi) \exp \left[-\frac{(x \pm \xi)^2}{4t} \right] d\xi, \quad x > 0, t > 0.$$

By differentiating inside the integral sign and splitting the range of integration at $x = \xi$ it is easily found that

$$\left| \frac{\partial J(x, t)}{\partial x} \right| < \frac{N}{(\pi t)^{1/2}},$$

where N is the upper bound of $f(x)$.

For (B_3) it is sufficient to prove that

$$(10) \quad J_1(x, t) \rightarrow 0 \quad \text{as } x \rightarrow +\infty,$$

$$(11) \quad J_2(x, t) - f(x) \rightarrow 0 \quad \text{as } x \rightarrow +\infty,$$

the approach being uniform in t in $0 < t \leq t_1$, where

$$J_1(x, t) \equiv t^{-1/2} \int_0^\infty f(\xi) \exp \left[-\frac{(x + \xi)^2}{4t} \right] d\xi, \quad x > 0, t > 0,$$

$$J_2(x, t) \equiv \frac{1}{2(\pi t)^{1/2}} \int_0^\infty f(\xi) \exp \left[-\frac{(x - \xi)^2}{4t} \right] d\xi, \quad x > 0, t > 0.$$

To prove (10) make the substitution $\xi = -x + 2\zeta t^{1/2}$ in J_1 :

$$|J_1(x, t)| \leq 2N \int_{x/2t^{1/2}}^\infty e^{-\zeta^2} d\zeta < \frac{4Nt_1^{1/2}}{x} \exp \left[-\frac{x^2}{4t_1} \right],$$

which clearly approaches zero as x approaches infinity, uniformly in t in $0 < t \leq t_1$.

To prove (11), we have

$$J_2(x, t) - f(x) = \frac{1}{2(\pi t)^{1/2}} \int_0^\infty \{f(\xi) - f(x)\} \exp \left[-\frac{(x - \xi)^2}{4t} \right] d\xi \\ + \frac{f(x)}{\pi^{1/2}} \int_{x/2t^{1/2}}^\infty e^{-\xi^2} d\xi, \quad x > 0, t < t \leq t_1.$$

In the first term of the right member, make the change of variable $\xi = x + 2\zeta t^{1/2}$. Then

$$(12) \quad |J_2(x, t) - f(x)| \leq \frac{3N}{\pi^{1/2}} \int_{x/4t^{1/2}}^\infty e^{-\zeta^2} d\zeta \\ + \frac{1}{\pi^{1/2}} \int_{-x/4t^{1/2}}^\infty |f(x) - f(x + 2\zeta t^{1/2})| e^{-\zeta^2} d\zeta.$$

Since $f(x)$ approaches a limit as $x \rightarrow \infty$, the second term of the right member of (12) can be made less than any $\epsilon > 0$ by choosing $x > X$, independent of t . The first term of the right member of (12) clearly approaches zero as required in order that (11) be true.

It may be remarked that the conditions imposed on $f(x)$ in Theorem 4 are merely sufficient, and not necessary. In particular, $f(x)$ need not have limits as $|x| \rightarrow \infty$. For example, $U(x, t) = x$ as defined by (6) satisfies conditions B if $f(x) = x$. However, in most physical applications $f(x)$ will satisfy the hypotheses of this theorem, since the problem considered here is an approximation, for small time values, to the finite composite solid problem.

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